MAT100 OVERVIEW OF CONTENTS AND SAMPLE PROBLEMS

MAT100 is a fast-paced and thorough tour of precalculus mathematics, where the choice of topics is primarily motivated by the conceptual and technical knowledge that is most frequently needed in later calculus courses, especially MAT103. *Note that we do not use calculators in any of our math classes.* Although they can be a useful analytic tool, they can also be an obstacle to learning in foundational classes like these where the aim is to master enough of the basic concepts and techniques to think independently through solving simpler problems that can be done by hand.

Basic Functions and their Transformations

Calculus courses like MAT103, 104, 175 and 201 are primarily example-based, and they assume a solid knowledge of basic functions and their graphs including $y = |x|$, $y = x^2$, $y = x^3$, $y = 1/x$ and $y = \sqrt{x}$. In MAT100, we quickly review these building blocks and then we learn how to represent standard transformations (such as reflections, vertical and horizontal shifts and rescaling) and determine the domain, range and graph of the transformed functions.

Example 1 (Absolute Values) The graph of $g(x) = 2 - 3|x + 1|$ can be obtained from the graph of $f(x) = |x|$ by a sequence of simple geometric transformations. What are the coordinates of the vertex of the graph of g? What is the slope of the right half of the graph of g? Where does the graph of g cross the x-axis? the y-axis? Sketch the graph of f and g on the same set of coordinate axes. What is the range of g?

Example 2 (Quadratic Functions) Complete the square to rewrite $f(x) = 2x^2 - 12x + 13$ in the form $a(x - b)^2 + c$. The graph of this function is a parabola. What are the coordinates of the vertex? Determine the domain and range of f as well as the x- and y-intercepts. Sketch the graph of f.

Example 3 (Other Parabolas) The equation $x + y^2 - 2y = 3$ defines a parabola $\mathcal{P}$ in the xy-plane. Solve this equation for y in terms of x. (Hint: Complete the square in y.) What is the vertex of $\mathcal{P}$? Where does it cross the axes? How it is related to the parabola $y = x^2$?

Example 4 (Rational Functions) If

$$f(x) = \frac{x - 1}{(x - 2}$$

then what is the domain of f? Where is $f(x) > 0$? Find a formula for x in terms of the parameter b if

$$\frac{x-1}{x-2} = b.$$

What does this tell us about the range of f? Sketch the graph of f.

Example 5 (Factoring Polynomials)

a) The polynomial $x^3 + 8$ has only one real root. Find it, and use this to factor this polynomial into a linear factor and an irreducible quadratic factor.

b) Use factoring to determine where the function

$$f(x) = \frac{x^3 - 1}{x^3 - 3x^2 - x + 3}$$

is positive. Give your answer in interval notation.

$\boxed{\textbf{Answers}}$

1. **(Absolute Values)** Vertex is at $(-1, 2)$. Slope is -3 when $x > -1$. The x-intercepts are $x = -5/3$ and $x = -1/3$. The y-intercept is -1. The range of g is $(-\infty, 2]$.

2. **(Quadratic Functions)** $f(x) = 2(x-3)^2 - 5$ is a parabola with vertex $(3, 5)$ from which it opens upward. The domain is $(-\infty, \infty)$ and the range is is $[5, \infty)$. The x-intercepts are $3 \pm \sqrt{5/2}$ and the y-intercept is 13.

3. **(Other Parabolas)** $y = 1 \pm \sqrt{3-x}$ tells us that the graph of C is the parabola $x = y^2$ reflected across the y-axis, shifted 3 units to the right and up 1 unit vertically so that its vertex will be at the point $(3, 1)$. It crosses the axes at the points $(0, 1+\sqrt{3})$, $(0, 1-\sqrt{3})$ and $(2, 0)$.

4. **(Rational Functions)** f is defined everywhere except $x = 2$ where the graph will have a vertical asymptote. It is positive for x in the intervals $(-\infty, 1)$ and also $(2, \infty)$. Solving $(x-1)/(x-2) = b$ for x gives $x = (2b-1)/(b-1)$ which is defined for every value of b except $b = 1$. The range of f is therefore all real numbers except 1.

5. **(Factoring Polynomials)** a) $x^3+8=(x+2)(x^2-2x+4)$ b) on the interval $(-\infty,-1)$ and on $(3,\infty)$.

Transcendental Functions and their Transformations

In addition to polynomial and rational functions and their inverses, this course reviews the basic properties of two important groups of transcendental functions, the trigonometric functions and the exponential functions. These functions arise in almost every aspect of both pure and applied mathematics. For non-mathematicians, they arise naturally whenever we model periodic behavior or exponential growth (where the rate of change of some quantity is proportional to its current amount). Their inverse functions are also needed since they tell us how to determine the control parameters (or inputs) that produce a desired output when working with these models. So MAT100 also reviews functions like $\arcsin x$ and $\arctan x$ as well as logarithmic functions.

The transcendental numbers π and e of course appear in connection to these topics. The number π is defined as the ratio of circumference to diagonal length in any circle, and so must certainly be part of any discussion of circles and other radially-symmetric geometric objects. Since the circle is used to define the periodic functions $\sin\theta$ and $\cos\theta$, the number π plays an important role in any discussion of periodic behavior involving these functions. The number e is often first encountered by students in discussions of compound interest where it appears as the limiting value of $(1+1/n)^n$ as n approaches infinity, but it appears in many other contexts as well.

As transcendental numbers, both π and e defy any attempts to represent them precisely in decimal or other fractional forms, but both are roughly equal to 3, with π a bit bigger than 3 and e a bit smaller. It is useful to know common approximations such as $\pi\approx 3.14$ and $e\approx 2.71$. It can also be useful to remember approximate values of the square roots that appear in basic trigonometry – most students will remember that $\sqrt{2}\approx 1.14$ and $\sqrt{3}\approx 1.73$.

Example 1 (Sorting Numbers) For each list place the given numbers in order from smallest to largest.

a) $\sqrt{12},\quad \sqrt[3]{8},\quad \dfrac{20}{8},\quad e,\quad \pi,\quad \dfrac{e}{\pi}$

b) $2\sin(\pi/3),\quad \tan(\pi/4),\quad \cos\pi,\quad \sec(\pi/4),\quad \cos(e),\quad \ln(\pi^3)$

Example 2 (Logarithmic Functions) If $f(x) = x\ln(x+2)$ then what is the domain of f? For what values of x will it be true that $f(x) > 0$? Give your answers as intervals.

Example 3 (Trigonometric Functions)

a) The equation $6\sin^2\theta - \sin\theta = 1$ has two solutions in the interval $[0, \pi]$. What are they?

b) Simplify the expression $\tan^2 x - \sec^2 x$ as much as possible.

Example 4 (Inverse Trigonometric Functions) Simplify

$$\frac{\pi - \arccos(1)}{\arcsin(1/2)}$$

as much as possible.

Example 5 (Exponential Functions) Find all points where the curve defined by the equation $8x^3 - 27y^3 + e^{xy} = 0$ crosses the x-axis. Find all points where the curve crosses the y-axis.

Answers

1. **(Sorting Numbers)**

   a) $e/\pi < 1 < \sqrt[3]{8} = 2 < 20/8 = 2.5 < e < \pi < \sqrt{12} = 2\sqrt{3}$

   b) We know that $\cos\pi = -1$, $\tan(\pi/4) = 1$, $\sec(\pi/4) = \sqrt{2}$, and $2\sin(\pi/3) = \sqrt{3}$. Since e lies between $\pi/2$ and π we know that e is a second quadrant angle and $\cos\pi < \cos(e) < \cos(\pi/2) = 0$. Finally, $\ln(\pi^3) = 3\ln\pi > 3$ because $\pi > e$ and so $\ln(\pi) > 1$. Thus

   $$\cos\pi < \cos(e) < \tan(\pi/4) < \sec(\pi/4) < 2\sin(\pi/3) < \ln(\pi^3).$$

2. **(Logarithmic Functions)** The domain is $(-2, \infty)$. The function will be positive if x is in the interval $(-2, -1)$ or in the interval $(0, \infty)$.

3. **(Trigonometric Functions)**

   a) $\theta_1 = \pi/6$ and $\theta_2 = 5\pi/6$

   b) -1

4. **(Inverse Trigonometric Functions)** 6.

5. **(Exponential Functions)** crosses the x-axis at $(0, 1/3)$. crosses the y-axis at $(-1/2, 0)$.

Geometry (Lines and Circles, Distance and Area)

Equations of lines play a fundamental role in MAT103 where one of the main topics is the slope of a curve at a point. Another important topic in MAT103 (leading into MAT104) is finding the area of general two-dimensional regions in the plane. In MAT100 we can deal with some simple starting cases.

Example 1 (Area, Circles, Lines and Triangles) Let L_1 be the line $y = 2x - 1$. The distance formula gives the length of part of L_1 connecting $(1, 1)$ and $(0, -1)$:

$$\text{Distance} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{1^2 + 2^2} = \sqrt{5},$$

and lets us write down the equation of the circle C_1 centered at $(1, 1)$ that passes through the point $(0, -1)$:

$$(x - 1)^2 + (y - 1)^2 = 5 \text{ or } x^2 - 2x + y^2 - 2y = 3.$$

Recall that the area of the circle is πr^2, or in this case 5π.

a) Find an equation of the form $Ax + By = C$ for the line L_2 passing through the point $(1, 1)$ and perpendicular to L_1. Where does L_2 cross the x-axis?

   Hint: Two lines of slope m_1 and m_2 will be perpendicular when $m_1 \cdot m_2 = -1$.

b) How many times will the circle C_2 centered at $(1, 1)$ and passing through the point $(3, 0)$ intersect the circle C_1 found above?

c) Consider the triangle T whose vertices are $(1, 1)$, $(3, 0)$ and $(0, -1)$. What is the area of T?

d) What is the area of the region that is bounded above by the line $x - 3y = 3$ and below by the circle C_1?

   Hint: Draw the picture.

Calculus Preview – Slope of a curve: Consider the line joining the points $(1,1)$ and a nearby point (b,b^2) on the parabola $y=x^2$. We can write down the slope of the line segment joining these points, and think about how the slope changes when we imagine sliding the point (b,b^2) along the parabola towards the point $(1,1)$:

$$\frac{\Delta y}{\Delta x} = \frac{b^2-1}{b-1} = \frac{(b+1)(b-1)}{b-1} = b+1 \to 2 \text{ as } (b,b^2) \to (1,1).$$

We can say that the slope of the parabola at the point $(1,1)$ will be 2 and the line of slope 2 passing through $(1,1)$ is called the tangent line to the parabola at that point. If we start at $(1,1)$ and move down 2 units and left 1 we will hit the point $(0,-1)$. So this line will have equation $y=2x-1$.

Example 2 (Tangent Line to a Parabola) Use the method outlined above to find the equation of the tangent line to the parabola $y=x^2$ at the point $(2,4)$.

Answers

1. **(Area, Circles, Lines and Triangles)**

   a) $x+2y=3$ and L_2 crosses the x-axis at the point $(3,0)$.

   b) The distance from $(1,1)$ to $(3,0)$ is also $\sqrt{5}$ so C_1 and C_2 are identical, and intersect infinitely many times.

   c) The area of the triangle will be $(1/2)\sqrt{5}\cdot\sqrt{5}=5/2$.

   d) The line $x-3y=3$ connects the points $(0,-1)$ and $(3,0)$. One radius of C_1 connects $(1,1)$ and $(3,0)$. Another radius connects the points $(1,1)$ and $(0,-1)$ and the angle between these two radii is $\pi/2$. So we take one fourth of the total area of the circle and throw away the part inside the triangle T to get an area of $5\pi/4-5/2$.

2. **(Tangent Lines to a Parabola)** $y=4x-4$