# MAT103 Overview and Sample Problems

All the important concepts in calculus involve limits and infinite processes and so we begin with an intensive but generally intuitive treatment of limits. (Rigorous treatment of limits is reserved for courses like 215.) Building on our knowledge of limits, we move on to the notions of continuity and differentiability. We learn to analyze common ways that continuity and differentiability may fail for a given function, and we develop a library of derivative formulas for all the standard functions which we can use to study rates of change, to understand the sensitivity of a system to small changes in the control parameters and as a tool for approximation. The overarching theme of the course is how to use limits and derivatives to understand the main features of the behavior of a complicated function. This theme is further developed through advanced curve-sketching and optimization in the second half of the course. Finally, we investigate cumulative change, average behavior and an introduction to modeling and simple differential equations when we learn about definite and indefinite integrals and the Fundamental Theorem of Calculus. These topics are further developed for functions of a single variable in Calculus II and in the multivariable setting in Calculus III.

All the sample problems here come from past MAT103 quizzes and exams and are chosen to represent core concepts and techniques from the class corresponding to a B-level of knowledge.

## Problems on Limits & Continuity

We use limits to understand general trends and important features related to the domain and range of complicated functions, especially near problematic points at the edge of the domain, where the input values or output values become infinitely large, or where we see competing trends of growth and decay in expressions that approach $0/0$ or $\infty/\infty$. *Students learn how to work without using a calculator as a substitute for a general working knowledge of the basic functions and their graphs.*

We need a variety of techniques to understand and compute limits (or to understand how exactly they may fail to exist), and often these general techniques must be modified or adapted depending on the special features of the functions that appear, requiring many ideas from precalculus including factoring techniques and other algebraic manipulations, properties of polynomial and rational functions and their inverses, the rules for manipulating logarithmic and exponential functions and also trigonometric functions and identities.

The notions of limit and continuity are closely related. We say that $f$ is continuous at a point if the behavior of $f$ there is consistent with the behavior of $f$ nearby:

$$\lim_{x\to c} f(x) = f\left(\lim_{x\to c} x\right) = f(c) \text{ means } f \text{ is continuous at } x = c.$$

**Example (Limit of type 0/0)** Consider the function

$$f(x) = \frac{3x^2 - 5x - 2}{x^3 - 8}$$

and its graph.

a) Factor $f(x)$ and simplify as much as possible.

b) Use your result in a) to compute the limit of $f(x)$ as $x \to 2$, or explain why this limit does not exist.

c) What does the calculation in part b) tell you about the graph of $f$?

**Note:** A problem like this would typically occur as an intermediate step in a question where we are asked to use limits to understand asymptotes and classify the discontinuities of a function.

Many students with prior calculus experience may be tempted to use l'Hôpital's Rule to compute the limit above. However, we will often encounter limits where l'Hôpital's Rule is not helpful. Instead we might need to understand the dominant behavior in different areas of the domain, a powerful general techniques which also plays a very important role in MAT104 to help us analyze the convergence behavior of infinite series and improper integrals.)

**Example (L'Hôpital Fails)** Consider the function

$$f(x) = \frac{2^x + 3^x}{3^x + 4^x}$$

and its graph.

a) Analyze the behavior of the numerator and the denominator as $x \to \pm\infty$. What is the "dominant term" in each case?

b) Use your work in a) to compute $\lim_{x\to\infty} f(x)$ or determine that this limit does not exist. What feature of the graph of $f$ does this limit calculation reveal?

c) Use your work in part a) to compute $\lim_{x\to-\infty} f(x)$ or determine that this limit does not exist. What feature of the graph of $f$ does this limit calculation reveal?

## Problems on Derivatives & Continuity

Building on our knowledge of limits, especially those of type $0/0$ we learn to compute the derivative $f'$ defined as

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \quad \text{or} \quad \lim_{z \to x} \frac{f(z) - f(x)}{z - x}$$

whenever this limit exists. The limit definition of derivative allows us to extend the idea of slope from straight lines to more general curves, and in the process we define the tangent line as the best linear approximation to a curve at a point.

**Example (Computing Derivatives)**

a) Find $dy/dx$ if $y = \tan^3(\ln x)$.

b) Find the slope of the curve $C$ defined by $xy = 3x + 2y$ at the points where it crosses the line $y = x$.

c) Find $b$ if the line connecting the points $(0, b)$ and $(1, 1)$ intersects the parabola $y = x^2$ in a right angle.

**Example (Rate of Change)** A particle moves along a straight line track so that its position at time $t \geq 0$ is given by

$$s(t) = (t^2 + 2t)e^{-t}.$$

Compute the velocity $v(t)$ of the particle. How far does the particle move forward before reversing direction? If this situation remains unchanged, what is the total distance the particle would travel during the time interval $0 \leq t < \infty$?

**Example (Continuity/Differentiability #1)** Suppose that

$$f(x) = x^2 - x - \alpha \text{ if } x \geq 0 \text{ and } f(x) = \beta x - 4 \text{ if } x < 0.$$

a) For what values of the parameter $\alpha$ will $f$ be continuous at $x = 0$? For what values of $\beta$ will the function $f$ be continuous at $x = 0$?

b) If the parameters $\alpha$ and $\beta$ are chosen in such a way that $f$ is continuous at $x = 0$, then for what values of $\alpha$ and $\beta$ will the function $f$ also be differentiable at $x = 0$?

**Example (Continuity/Differentiability #2)** Suppose that

$$f(x) = x \sin(1/x) \text{ if } x > 0 \text{ and } f(x) = 0 \text{ if } x \leq 0.$$

Compute the necessary limits to determine whether or not $f$ is continuous at $x = 0$. Use the limit definition of the derivative to determine whether or not $f$ is differentiable at $x = 0$.

## Problems on Optimization, Extrema, Curve-Sketching

Once we have learned how to compute the slope of a curve we can analyze the sign of the slope function to understand the general features of the curve. We can determine the range of quite complicated functions and locate the peaks and valleys in their graphs with precision (provided we can solve for the roots of the slope function, often the most difficult part of the problem). Understanding the (local and global) extreme values of a function is the mathematical foundation for the theory of optimization, one of the most frequent and important applications of calculus, allowing us to understand how to choose input or control parameters to maximize good outcomes and minimize bad ones.

Constrained optimization is important in many disciplines, both as a theoretical tool and also in applications. This topic is further developed in MAT201 (and also in MAT175), but already in MAT103 we can solve some interesting examples. These problems often require that we are familiar with geometric formulas for area and volume including area of triangles and circles as well as surface area and volumes of spheres, cylinders and cones. Given constraints in the form of construction materials of a particular type, we can use calculus as one tool in making design decisions, for example to design a container with maximum storage capacity and minimum construction cost.

**Example (Curve-Sketching)** Consider the function $f(x) = x^2 \ln x$. What is its natural domain? Determine the angle with which this curve crosses the $x$-axis. Where is $f$ increasing? decreasing? Determine all local and global extrema for $f$ or determine that they do not exist. Determine the range of $f$.

**Example (Constrained Optimization)** A rectangular sheet is to be rolled into a hollow cylinder by gluing a par of opposite sides together. Given that the sheet has perimeter $6\pi$ centimeters, what is the maximum possible volume of the cylinder obtained in this way?

## Problems on Integration

The last big topic in MAT103 is integration which is further developed in MAT104 and which is extended to more complicated regions in three dimensional space in the second half of MAT201. The definite integral $\int_a^b f(x)\,dx$ defines the area under a continuous curve on an interval $[a, b]$ as the limit of approximating sums. The Fundamental Theorem of Calculus connects integration to what we have learned about derivatives and rates of change, giving an efficient way to compute these areas, provided we can find an antiderivative for the integrand $f$.

**Reminder:** The definite integral counts signed area, and thus the definite integral over a region where $f(x)$ changes sign will not represent our usual notion of the area, due to cancellation. We use the term *total area* to refer to the more conventional idea of area where we adjust to eliminate this cancellation effect.

Frequently we can work backwards from theoretical or conjectural information about rates of change to infer a model that describes all states of our system. Projectile motion is a good starting example of this kind of analysis. Given that gravitational acceleration increases velocity toward the surface of the earth at a rate of (roughly) 10 meters per second for every second of an object's free fall (in the absence of significant air resistance), we can work backwards from $a(t) = -10$ ft/sec/sec to infer that

$$v(t) = v_0 - 10t \text{ ft/sec and } h(t) = h_0 + v_0 t - 5t^2 \text{ ft}$$

where $v_0$ is the initial velocity and $h_0$ is the initial height at time $t = 0$ seconds.

Geometrically, if we have a reasonable formula for the slope of a curve as a function of $x$ and we have a starting point $(x_0, y_0)$ on the curve, then we can find a formula that describes the curve itself by finding an antiderivative for the slope function and adjusting the constant of integration to make sure our curve passes through the given starting point. This introduces the rather big subject of differential equations, one of the most important topics in both pure and applied mathematics. For students in MAT103 it is a good way to practice finding antiderivatives. Students in MAT104 learn several powerful techniques for computing antiderivatives and then apply those skills to additional examples of elementary differential equations that appear commonly in many introductory science and engineering courses.

**Example (Finding Area):** Set up a definite integral for the total area of the finite region bounded by the line $y = x$ and the curve $y = x^3 - 3x$.

**Example (Projectile Motion):** A tennis ball is dropped (not thrown) from a building that is 80 meters tall. How long will it take the tennis ball to hit the ground? What will its speed be when it hits the ground? The tennis ball loses some energy in the impact and bounces back up with 75% of its impact speed. How high will the tennis ball go on its first bounce? What is the average speed of the tennis ball during the time interval $0 \leq t \leq 10$?

**Example (Initial Value Problem)** If

$$dy/dx = x\sqrt{4 - x^2}$$

and $y(0) = 0$ then find a formula for $y(x)$.

# Answers

1. **(Limit of type 0/0)**

   a) $\dfrac{3x^2-5x-2}{x^3-8} = \dfrac{3x+1}{x^2+2x+4}$

   b) $7/12$

   c) $f$ has a removable discontinuity at $x=2$; its graph has a 'hole' at the point $(2, 7/12)$.

2. **(L'Hôpital Fails)**

   a) $3^x$ dominates on top and $4^x$ dominates on the bottom as $x \to +\infty$. $2^x$ dominates on top and $3^x$ dominates on the bottom as $x \to -\infty$.

   b) $\lim_{x\to\infty} f(x) = 0$ and $y=0$ is a horizontal asymptote as $x \to \infty$.

   c) $\lim_{x\to-\infty} f(x) = \infty$. As $x$ goes to $-\infty$, the graph is asymptotic to $y = (2/3)^x$.

3. **(Computing Derivatives)**

   a) $\dfrac{dy}{dx} = \dfrac{3\tan^2(\ln x)\sec^2(\ln x)}{x}$

   b) $y' = (3-y)/(x-2)$ and there are two intersection points: $(0,0)$ and $(5,5)$. The slope of $C$ at $(0,0)$ is $-3/2$ and the slope of the curve at $(5,5)$ will be $-2/3$.

   c) The slope of the parabola at $(1,1)$ is 2, so we need the line joining $(0,b)$ and $(1,1)$ to have slope $-1/2$. So $b = 3/2$.

4. **(Rate of Change)** $v(t) = e^{-t}(2-t^2)$. The particle moves forward when $0 \le t < \sqrt{2}$, reaching position $s(\sqrt{2}) = \frac{2+2\sqrt{2}}{e^{\sqrt{2}}}$ when it reverses direction and heads back toward its original position. Since $s(t) \to 0$ as $t \to \infty$, the particle will travel a total distance of $2s(\sqrt{2}) = \frac{4+4\sqrt{2}}{e^{\sqrt{2}}}$ during the given (infinite) time interval.

5. **(Continuity/Differentiability #1)** For continuity we must have $\alpha = 4$, but $\beta$ can be any real number. For differentiability, we must have $\alpha = 4$ and $\beta = -1$.

6. **(Continuity/Differentiability #2)** $f$ is continuous because

$$f(0) = 0 = \lim_{x \to 0^+} x \sin(1/x)$$

by the Sandwich or Squeeze Theorem. $f$ is not differentiable there because the right-hand derivative

$$\lim_{h \to 0^+} \frac{h \sin(1/h)}{h} = \lim_{h \to 0^+} \sin(1/h) = \lim_{u \to \infty} \sin(u)$$

does not exist, due to oscillation.

7. **(Curve-Sketching)** The natural domain is $(0, \infty)$. The graph crosses the $x$-axis at a $45^o$ angle because the slope there is 1. There is local minimum at the point $(1/\sqrt{e}, -1/(2e))$. The function is increasing on the interval $(1/\sqrt{e}, \infty)$ and decreasing on the interval $(0, 1/\sqrt{e})$, There is no global maximum and the range is $[-1/(2e), \infty)$.

8. **(Constrained Optimization)** Maximum volume possible is $\pi^2$, obtained from a $\pi \times 2\pi$ rectangle.

9. **(Finding Area)** $2 \displaystyle\int_0^2 (4x - x^3)\, dx = 8.$

10. **(Projectile Motion)** The tennis ball hits the ground after 4 seconds with a speed of 40 m/sec. It bounces back up with a speed of 30 m/sec and reaches its peak height of 45 m after 3 seconds. The total distance traveled by the tennis ball in the first 10 seconds is $80 + 45 + 45$, and so the average speed is 17 m/sec.

11. **(Initial Value Problem)** $y(x) = \dfrac{8}{3} - \dfrac{(\sqrt{4 - x^2})^3}{3}$