MAT104 Overview and Sample Problems

The first main topic of MAT04 is integration, making up about a third of the course. We learn standard techniques for computing integrals, such as integration by parts and substitution, making use of a wide variety of techniques and tools such as factoring, completing the square, polynomial division and other algebraic tools while also working with rational functions, trigonometric functions (and their inverses) as well as logarithms and exponentials.

Even more important than computing integrals or transforming them to equivalent forms is the ability to set up an integral and understand what it represents in various contexts. We develop this skill through problems in geometry, describing the area of regions in the plane, length of planar curves, volumes and surface areas for some special types of regions in 3-space. (This work is expanded and continued in the second half of MAT201, where we develop the basic tools needed to do calculus in higher dimensions.)

Combining what we know about limits and rates of growth to what we have learned about integrals and area leads to an interesting new notion: convergence and divergence of improper integrals. Building on our work in MAT103 working with limits and linear approximation, we study infinite series and, in particular, Taylor series, in MAT104. We further develop our ability to recognize and quantify rates of growth and learn how to develop simplifying approximations that capture the essence of a mathematical description, leading to insights or allowing efficient numerical estimates when technical obstacles block exact calculations.

In both the biological and the physical sciences, differential equations are used to build theories and make predictions. An introduction to this topic makes up another third of MAT104, and in the process we make good use of our integration skills, we learn the basics of complex numbers and we review topics like polar coordinates which will be much used in MAT201.

All the sample problems here come from past MAT104 quizzes and exams and are chosen to represent core concepts and techniques from the class corresponding to a B-level of knowledge.

Problems on Computing Integrals

Example (Area & Volume): Consider the region R between the curve $y = \sin x$ and the line $y = x$ for $0 \leq x \leq \pi$. Compute the area of R. Find the volume of the surface S_1 obtained by rotating R about the x-axis. Find the volume of the surface S_2 obtained by rotating R about the y-axis. Estimate your answers to the nearest integer (without using a calculator).

Example (Arc Length & Surface Area) Consider the arc $\mathcal{A}$ joining the point $(0,0)$ and $(1,1)$ along the curve $y^2 = x^3$. Find the length of $\mathcal{A}$. Set up a definite integral for the area of the surface $\mathcal{S}$ obtained by rotating $\mathcal{A}$ about the y-axis.

Example (Antiderivative) Find an antiderivative $F(x)$ for

$$f(x) = \frac{1}{\sqrt{x^2 - 1}}.$$

Example (Improper Integral #1) Compute $A(b)$ defined as the area under the curve $y = 1/(x^2 - 1)$ for x in the interval $[2, b]$. Compute the limit of $A(b)$ as $b \to \infty$.

Problems on Convergence & Divergence

Remember that we call an improper integral convergent when the limiting value exists and is finite. Otherwise, the integral is called divergent. We may be able to determine whether an integral converges or diverges by finding an antiderivative, applying the Fundamental Theorem of Calculus, and then taking a limit (as the example above illustrates). More frequently, finding an antiderivative is too difficult. In such cases, we analyze rates of growth or decay as the integrand approaches a vertical or horizontal asymptote and make comparisons to simpler integrals in order to determine convergence or divergence indirectly. Similar arguments apply to infinite series.

Example (Improper Integral # 2) Determine whether the improper integral $\displaystyle\int_2^\infty \frac{dx}{\sqrt{x^2 - 1}}$ converges or diverges. Justify your answer. Determine whether the improper integral $\int_2^{\infty} \frac{dx}{\sqrt{x^3-1}}$ converges or diverges. Justify your answer.

Example (Geometric Series) Compute the sum $\sum_2^{\infty} \frac{2^n}{3^{n+1}}$ or show that this series diverges.

Example (Series Convergence) Which of the following series converge? Justify your answers!

$$\sum_{n=0}^{\infty} \frac{n^2}{\sqrt{n^5+1}} \qquad \sum_{n=2}^{\infty} \frac{1}{n \ln n} \qquad \sum_{n=1}^{\infty} \left(\frac{n-1}{n}\right)^n \qquad \sum_{n=1}^{\infty} \left(\frac{n-1}{n}\right)^{n^2}$$

Problems on Power & Taylor Series

Power series are a special type of infinite series that allow us to define a function f as the sum of a series whose individual terms depend on x. These series are very important from a theoretical point of view, but they are also very useful in applications because they give us a sequence of more and more accurate polynomial approximations to transcendental functions involving e^x, $\ln x$, $\sin x$, etc.

Example (Interval of Convergence) Determine the interval of convergence for the power series

$$\sum_{n=2}^{\infty} \frac{(x-2)^n}{n \ln n}.$$

Taylor series are a generalization of the tangent line approximation to $y = f(x)$ at a point $(a, f(a))$ that makes use of the higher derivatives of an infinitely differentiable function to create a sequence of polynomial approximations of increasing accuracy (under the right conditions). The most important Taylor series should be memorized, along with the intervals where they are valid!

Example (Standard Taylor Series) Construct or recall the Taylor series for e^x, for $\sin x$ and for $\cos x$. Note that these series are valid for all real values of x.

New Taylor Series can be computed from known ones by making substitutions and other manipulations like differentiating or integrating. (The latter operations won't affect the interval of convergence except perhaps at the endpoints.)

Example (Contructing Taylor Series by Substitution) Determine the Taylor series for $g(x) = xe^{2x}$ and for $h(x) = \sin(x^2)$ from the known series for e^x and for $\sin x$.

Example(Differentiating Taylor Series) Use the geometric series expansion

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^2$$

to construct the Taylor Series expansion for $g(x) = 1/(1-x)^2$. Where does this series converge?

Example (Limits From Taylor Series) Use Taylor Series to compute

$$\lim_{x\to 0} \frac{\cos(x^2) - 1 + x^4/2}{x^2(x - \sin x)^2}$$

Example (Improper Integral From Taylor Series) Use Taylor series to analyze the convergence behavior of

$$\int_0^1 \frac{\sqrt{\sin x}}{1 - \cos x}\, dx.$$

Problems on Complex Numbers & Differential Equations

Example (Complex Numbers) Find all solutions $z = x + iy$ that satisfy the equation

$$|iz - 1| = |z - 1|.$$

Example (Complex Roots) Find all roots (real and complex) of the equation $z^4 = -4$.

Example (Linear First-Order DiffEq) Find all functions $y(x)$ satisfying

$$x\frac{dy}{dx} + y = x \ln x \text{ and } y(1) = 1.$$

Example (Linear Second-Order DiffEq) Find $y(t)$ if

$$y''(t) - 4y'(t) + 8y(t) = 0$$

and the graph of y crosses the y-axis at $y = 4$ with slope -2.

Answers

1. **(Area & Volume)** The area of R is $\pi^2/2 - 2$ (≈ 3). The volume of S_1 is $\pi^4/3 - \pi^2/2$ (≈ 27). The volume of S_2 is $2\pi^4/3 - 2\pi^2$ (≈ 45).

2. **(Arc Length & Surface Area)** The length will be $(13\sqrt{13} - 8)/27$. The surface area can be computed as the integral of $2\pi x\sqrt{1 + 9x/4}$ for $x \in [0, 1]$. (Other answers are possible, but you should be able to show that your answer is equivalent to this.)

3. **(Antiderivative)** The trigonometric substitution $x = \sec\theta$ leads to an antiderivative of the form $\ln|\sec\theta + \tan\theta|$ which translates to $F(x) = \ln|x + \sqrt{x^2 - 1}|$.

4. **(Improper Integral #2)** The first integral diverges, either by direct computation or by limit comparison with $1/x$. The second integral converges by limit comparison with $1/x^{3/2}$.

5. **(Geometric Series)** The series converges to $4/9$.

6. **(Series Convergence)** The first series diverges by limit comparison to $1/\sqrt{n}$. The second series diverges (slowly) by the integral test. The third series diverges by the n-th term test since the n-th term of this series approaches $1/e$ as $n \to \infty$. The fourth series converges by the root test.

7. **(Interval of Convergence)** The series converges for x in $[1, 3)$.

8. **(Standard Taylor Series)**

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\sin x = x - \frac{x^3}{3!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\cos x = 1 - \frac{x^2}{2!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

9. **(Contructing Taylor Series by Substitution)**

$$g(x) = xe^{2x} = x + 2x^2 + \frac{2^2x^3}{2!} + \cdots + \frac{2^n x^{n+1}}{n!} + \cdots$$

$$h(x) = \sin(x^2) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{(2n+1)!}$$

10. **(Differentiating Taylor Series)**

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \cdots + nx^{n-1} + \cdots$$

holds on the interval $(-1, 1)$.

11. **(Limits From Taylor Series)** The limit is 3.

12. **(Improper Integral From Taylor Series)** The only issue is the behavior as $x \to 0$. The integrand behaves like $1/\sqrt{x^3}$ and so the integral diverges by the p-test.

13. **(Complex Numbers)** The solutions will be all complex numbers of the form $x - ix$, where x can be any real number.

14. **(Complex Roots)** There are four solutions: $1 + i$, $1 - i$, $i - 1$ and $-i - 1$.

15. **(Linear First-Order DiffEq)** There is a unique solution:

$$y(x) = \frac{x \ln x}{2} - \frac{x}{4} + \frac{5}{4x}$$

16. **(Linear Second-Order DiffEq)** There is a unique solution:

$$y(t) = (4\cos(2t) - 5\sin(2t))e^{2t}$$