MAT175 Overview and Sample Problems

The course begins with a quick review/overview of one-variable integration including the Fundamental Theorem of Calculus, u-substitutions, integration by parts, and partial fractions in the context of studying integrals involving infinity, the logistic equation and area of regions with curved boundary in the xy-plane. (These topics are covered more thoroughly in MAT104.)

Next we learn how to work with vectors in 2- and 3-space along with algebraic tools like the distance and midpoint formulas, the dot product and the cross-product. We look at general curves in higher dimensions, mostly focused on lines and tangent lines in 3-space. We move on to discuss planes in 3-space and in this context we develop some basic tools from linear algebra including a brief, mostly computational, treatment of determinants, matrices and Gaussian elimination, all in preparation for learning calculus for functions of 2 or more variables. We continue our study of surfaces in 3-space with a thorough treatment of quadric surfaces (spheres, ellipsoids, hyperboloids, paraboloids, cylinders, cones) as well as hyperbolic paraboloids like the surface defined by $z = x^2 - y^2$.

Building on this foundation, we extend the ideas of limit, continuity and differentiability to functions of 2 or more variables. Relying heavily on a solid foundation from differential calculus in a single variable (MAT103) we learn about how to analyze a surface obtained as the graph of a function of two variables as well as surfaces obtained as a level set of a function of three variables. We look at how to compute limits and how to describe tangent planes and rates of change via partial derivatives, directional derivatives and the gradient. We learn how to locate the extrema of functions of two or more variables and how to recognize saddle points. Constrained optimization and Lagrange Multipliers are the main application and this topic is the culmination of our work with derivatives in higher dimensions.

Finally, we introduce the definite integral over regions in 2- or 3-space and briefly discuss how to compute volumes using iterated integration.

Note: All the sample problems here come from past MAT175 quizzes and exams and are chosen to represent core concepts and techniques from the class corresponding to a B-level of knowledge.

Example (Integration By Parts) Set up and compute a definite integral to find the area of the region between the curve $y = x^2 \ln x$ and the x-axis where $1 \leq x \leq 2$.

Example (Improper Integral & Partial Fractions) Find a function $F(x)$ so that

$$F'(x) = \frac{5}{2x^2 + x - 3}$$

and set up an improper integral that represents the area under the curve $y = 5/(2x^2 + x - 3)$ to the right of the vertical line $x = 2$. Compute this area if it is finite, or explain why it is infinite.

Example (Logistic Equation) a) Suppose that a group of 100 zombies arrive in Princeton among the class of 2022. Based on past zombie outbreaks we know that in the early stages of the epidemic the spread of infection is governed by the differential equation $dz/dt = 10^4 z$ where t is measured in months. Solve this differential equation for z as a function of t. b) Further study of zombie outbreaks reveals that as the number of zombies increase in an isolated population like Princeton University, the rate of growth dz/dt will decrease over time as the overall zombie chaos grows. Therefore a better model would be the logistic equation

$$\frac{dz}{dt} = 10z(1000 - z) = 10^4 z - 10z^2$$

which predicts that the zombie population will level off and approach some finite value Z_{cap}, the zombie capacity. What is the numerical value of Z_{cap} for this logistic equation? **Note:** We learned how to solve the logistic equation explicity for z but it is not necessary to do that in order to find Z_{cap}. Just be sure to explain your reasoning.

Example (Vectors and 3-space) a) Find the center and radius of the sphere given by the equation

$$x^2 + y^2 + z^2 - 4x - 6y + 10z + 13 = 0.$$

b) Find the point D so that $ABCD$ is a parallelogram where

$$A = (1, -2, 4) \quad B = (2, 1, 6) \quad C = (-1, -2, 5)$$

and D is diagonally opposite to B.

Example (Dot Product, Projections) Three points $P = (1, 1)$, $Q = (7, 3)$ and $R = (2, 8)$ are given in the xy-plane.

a) Find the vector $\overrightarrow{PQ}$.

b) Find the angle θ between vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$.

c) Find the *vector* projection of $\overrightarrow{PR}$ onto the vector $\overrightarrow{PQ}$.

d) Find two vectors $\vec{u}$ and $\vec{v}$ such that $\overrightarrow{PR} = \vec{u} + \vec{v}$ where $\vec{u}$ is parallel to $\overrightarrow{PQ}$ and $\vec{v}$ is perpendicular to $\overrightarrow{PQ}$.

e) Find the area of the triangle PQR.

Hint for parts d) and e): On the same xy-plane, draw triangle PQR and the projection $\text{Pr}_{\overrightarrow{PQ}}(\overrightarrow{PR})$ (from part (c)) starting from point P.

Example (Cross-Product) Consider the three points $A(2,3,1)$, $B(-1,0,2)$ and $C(-2,4,1)$ in 3-space. Find the equation of the plane passing through these three points. Find the area of the triangle with points A,B,C as vertices.

Example (Planes & Lines) Given that line L_1 is defined by the equations

$$\frac{x-2}{3} = \frac{y+1}{1} = \frac{z+3}{-2},$$

and line L_2 is defined by the equations

$$x = -1 + 4s, \quad y = 3 - 2s, \quad z = 4 - 6s,$$

it can be shown that L_1 and L_2 intersect at a point. Find an equation of the plane that contains both L_1 and L_2.

Example (Intersection of Lines in 3-space) Let L_1 be the line given parametrically by

$$x = 2 - 3t, \quad y = 4 + t, \quad z = -t$$

and L_2 the line given by

$$x = 1 + s, \quad y = 2 + 6s, \quad z = -1 + s.$$

Find the point of intersection of L_1 and L_2 or show that the lines are in skew position.

Example (Curvilinear Motion) Let $\mathbf{r}(t) = \frac{1}{3}t^3\mathbf{i} + e^{-2t}\mathbf{j} + 5\ln(t-1)\mathbf{k}$ describe the position of a particle moving in 3-space. What is the natural domain of the function $\mathbf{r}(t)$? What is the velocity of the particle at time $t = 2$? Is the particle moving upward or downward at that time? (Justify your answer.) Find the acceleration at time $t = 2$. At this time is the angle between the velocity vector and the acceleration vector acute or obtuse? Find parametric equations for the tangent line to the path of the particle at time $t = 2$.

Example (Surfaces) The equation $y^2/9 + z^2/4 = x^2 + 1$ defines a surface. Draw the trace in the xz-plane. Label the intercepts with the positive coordinate axes. Draw the trace in the yz-plane. Label the intercepts with the positive coordinate axes. Draw the surface defined by this equation.

Example (Function of 2-variables/Level Curves) Consider the function $f(x, y) = 9 - x^2/4 - y^2$. Draw the level sets at level 0 and 5 on the same set of axes. Draw the trace for the graph of f in the xz-plane. Draw the trace of the graph of f in the yz-plane. Draw the graph of f and label the points where the graph intersects the positive x-axis, the positive y-axis, and the positive z-axis. What is the range of f?

Example (Limit) If

$$f(x, y) = \frac{x^2y - xy}{x^3 + xy^2 - x^2 - y^2}$$

show that $\lim_{(x,y)\to(0,0)} f(x, y)$ does not exist.

Example (Partial Derivatives) Let $f(x, y) = x^2 \ln(xy^2)$. Draw the domain of f (and explain your reasoning). Calculate and simplify

$$f(1, e), \quad f_x(1, e), \quad \text{and } f_y(1, e).$$

Let C be the curve of intersection between the graph of f and the plane $x = 1$. Find parametric equations for the curve C. Let T be the tangent line to C at the point $(1, e, f(1, e))$. Find parametric equations for T.

Example (Directional Derivatives) Let $f(x, y) = \sqrt{9 - x^2 - y^2}$. Describe the surface formed by the graph of f in words and sketch it, labeling the points where the graph intersects the positive coordinate axes. Find the directional derivative of f at the point $P = (1, 2)$ in the direction $\langle 1, 1\rangle$. Find an equation for the tangent plane to the graph of f at the point where it intersects the planes $x = 1$ and $y = 2$. Give your answer in the form $Ax + By + Cz = D$ where A, B, C and D are all integers.

Example (Chain Rule/Implicit Differentiation) Consider the function $z = f(x, y)$ defined implicitly by the equation

$$6e^z y^2 - 3x\cos(z) + x = 0$$

with the additional property that $f(3, 1) = 0$. Compute $\dfrac{\partial z}{\partial x}(3, 1)$.

Example (Tangent Plane) Find the point P on the surface S defined by $2z = 3x^2+2y^2$ where the tangent plane is parallel to the plane $3x-4y+z = 0$. Let C be the curve of intersection of the surface S with the plane $x-5y+2z = 0$, a plane that also contains the point P. Find parametric equations for the tangent line to C at the point P.

Example (Extrema) Consider the function $f(x, y) = 9xy - x^3 - y^3$. Find all critical points. For each, determine whether it gives a local minimum, local maximum or saddle point. Then restrict f to the closed and bounded triangular region T with vertices $(0, 0)$, $(1, 0)$ and $(1, 2)$. Analyze the global extrema of f on this restricted domain.

Example (Lagrange) Use the Lagrange multiplier method to find the dimensions of the open rectangular box (open = no lid) whose surface area (and hence manufacturing cost) is as small as possible given that we require its volume to be 108 cubic centimeters.

Example (Double Integral) Evaluate the double integral $\iint_S e^x y\, dA$ where S is the triangular region with vertices $(0, 0)$, $(1, 1)$, and $(0, 2)$.

Example (Linear Algebra)

a) Use Gauss-Jordan elimination to find parametric equations for the line of intersection between the planes

$$\begin{aligned} x + y + 5z &= 5 \\ 3x + 2y + 4z &= 1 \end{aligned}$$

At each step, indicate which row operation is being performed.

b) Use any method to solve the system of equations:

$$\begin{aligned} x + y + 5z &= 5 \\ 3x + 2y + 4z &= 1 \\ -x - 3y + 3z &= 7 \end{aligned}$$

Answers

1. **(Integration by Parts)**

$$\int_1^2 x^2 \ln x \, dx = \frac{8}{3}\ln 2 - \frac{7}{9}$$

2. **(Improper Integral & Partial Fractions)**

$$F(x) = \frac{1}{x-1} - \frac{2}{2x+3}$$

and the area we are asked for is finite, given by

$$\int_2^\infty \frac{5}{2x^2+x-3}\, dx = \ln 7 - \ln 2.$$

3. **(Logistic Equation)** For part a) we have simple exponential growth and $z(t) = 100e^{10^4 t}$. For the logistic growth in b) Z_{cap} will be 1000.

4. **(Vectors and 3-space)**

   a) Center is $(2, 3, -5)$ and radius is 5. b) $D = (-2, -5, 3)$

5. **(Dot Product, Projections)** a) $\langle 6, 2\rangle$ b) $\theta = \arccos(1/\sqrt{5})$

   c) $\langle 3, 1\rangle$ d) $\vec{u} = \langle 3, 1\rangle$ and $\vec{v} = \langle -2, 6\rangle$

   e) The area of the triangle is 20.

6. **(Cross-Product)** The plane is $x + 4y + 15z = 29$ and the area of the triangle is $11/\sqrt{2}$.

7. **(Planes & Lines)** $y = x + z$.

8. **(Intersection of Lines in 3-space)** The lines are in skew position.

9. **(Curvilinear Motion)** The natural domain for t in $(1, \infty)$. The velocity vector is $\langle 4, -2/e^4, 5\rangle$, and the motion is upward since the z-component of the velocity vector is positive. The acceleration vector is $\langle 4, 4/e^4, -5\rangle$ and the angle is obtuse. The tangent line to the curve is given parametrically by $x = 4t + 8/3$, $y = 1/e^4 - 2t/e^4$, and $z = 5t$.

10. **(Surfaces)** The trace in the xz-plane is the curve $z^2/4 - x^2 = 1$, a hyperbola opening along the z-axis with vertices at $(0, \pm 2)$. The trace in the yz-plane is the curve $y^2/9 + z^2/4 = 1$, an ellipse centered at the origin, crossing the y-axis at ± 3 and crossing the z-axis at ± 2. The surface is a single-sheeted hyperboloid whose cross-sections along the x-axis are ellipses.

11. **(Function of 2-variables/Level Curves)** If $z = 0$ we get the ellipse $x^2/36 + y^2/9 = 1$ and if $z = 5$ we have another ellipse $x^2/16 + y^2/4 = 1$. The trace in the xz-plane is a parabola $z = 9 - x^2/4$ opening downward from its vertex at $(0, 9)$. The graph is an elliptic paraboloid crossing the positive z-axis at 9, crossing the positive x-axis at 6 and crossing the y-axis at 3. The range of f is $(-\infty, 9]$.

12. **(Limit)** If we approach $(0, 0)$ along the line $y = mx$, we get a limiting value of $m/(1 + m^2)$. We see that the limit depends on the direction we approach along. For example, if we approach along $y = x$, the limit will be $1/2$ and if we approach along $y = -x$, the limit will be $-1/2$.

13. **(Partial Derivatives)** The domain is the right half plane with $x > 0$ and $y \neq 0$.

$$f(1, e) = 2 \quad f_x(1, e) = 5 \quad f_y(1, 3) = 2/e.$$

The curve C is given parametrically by $x = 1$, $y = t$ and $z = \ln(t^2)$. The line T is given parametrically by $x = 1$, $y = e + s$ and $z = 2 + 2s/e$.

14. **(Directional Derivatives)** The surface is the top half of the sphere of radius 3 centered at $(0, 0, 0)$. The directional derivative is $-3/(2\sqrt{2})$. The tangent plane has equation $x + 2y + 2z = 9$.

15. **(Chain Rule/Implicit Differentiation)** $\partial z/\partial x = 1/3$ when $x = 3$ and $y = 1$.

16. **(Tangent Plane)** $P = (-1, 2, 11/2)$ and the line is given parametrically by

$$x = 3t - 1, y = 5t + 2, z = 11t + 11/2.$$

17. **(Extrema)** There are two critical points $(0, 0)$, a saddle point, and $(3, 3)$, a local maximum. On the triangle, the global min is -1, and it occurs at the point $(1, 0)$. The global maximum is $6\sqrt{3} - 1$ which occurs at $(1, \sqrt{3})$.

18. **(Lagrange)** $x = y = 6$ centimeters and $z = 3$ centimeters.

19. **(Double Integral)** $2e - 4$

20. **(Linear Algebra)** a) $x = 6t - 9$, $y = 14 - 11t$ and $z = t$, a free variable. b) $x = -1$, $y = -2/3$, $z = 4/3$