MAT203 OVERVIEW OF CONTENTS AND SAMPLE PROBLEMS

MAT203 covers essentially the same material as MAT201, but is more in depth and theoretical. Exam problems are often more sophisticated in scope and difficulty level.

All sample problems here come from past MAT203 quizzes and exams and are chosen to represent core concepts and techniques from the class corresponding to a B-level of knowledge.

Problems on Vectors and Basic Geometric Objects in $\mathbb{R}^3$

Example 1 (Vector Operations)

(a) Suppose $\vec{u}$ and $\vec{v}$ are two vectors in $\mathbb{R}^n$. If $\|\vec{u}+\vec{v}\| = \|\vec{u}-\vec{v}\|$, prove that $\vec{u}$ and $\vec{v}$ must be orthogonal.

(b) Let $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$ be three non-coplanar vectors. For any vector $\vec{x} \in \mathbb{R}^3$, there is a unique way to write $\vec{x} = a\vec{u} + b\vec{v} + c\vec{w}$ for some real numbers a, b, c. Find expressions for a, b, c in terms of $\vec{u}, \vec{v}, \vec{w}$.
Hint: Take dot product with $\vec{v} \times \vec{w}$.

Example 2 (Lines and Planes)

Consider two lines

$$L_1 : \qquad \langle 1+2t, -1+4t, 2+3t \rangle$$

$$L_2 : \qquad \langle 1-t, 1+2t, 4t \rangle$$

(a) Show that the two lines are in skew position.

(b) Find the distance between the two lines.

Example 3 (Curves and Quadric Surfaces)

Suppose $(x, y, z) \in \mathbb{R}^3$.

(a) Use the Cauchy-Schwarz inequality to prove that

$$xy + yz + zx \le x^2 + y^2 + z^2$$

(b) Let S be the surface in $\mathbb{R}^3$ given by

$$x^2 + y^2 + z^2 - xy - yz - zx = 1,$$

Note that the origin $O = (0,0,0)$ is not on S. Find the parametric equation of a line L_1 through the origin O such that L_1 *never intersects* S.

Hint: Note that

$$2(x^2 + y^2 + z^2 - xy - yz - zx) = (x-y)^2 + (y-z)^2 + (z-x)^2.$$

(c) Note that the point $P = (1,0,0)$ is on S. Let line L_2 be the line through P with the same direction as L_1. Show that L_2 lies *entirely* on S.

(d) Find the distance between lines L_1 and L_2.

(e) Find an equation of the plane Π through the origin O and perpendicular to the line L_1.

(f) Let curve C be the intersection of the plane Π with the surface S. Show that C is a circle and centered at the origin and find its radius.

Hint: Can you combine the equations for Π *and* S *to a single equation involving only* x *and* y*? Can you express the distance from points* (x, y, z) *on* C *to the origin as an expression involving only* x *and* y*?*

Answers

1. **(Vector Operations)**

   (a) Expand $\|\vec{u} \pm \vec{v}\|^2 = (\vec{u} \pm \vec{v}) \cdot (\vec{u} \pm \vec{v})$.

   (b)

$$a = \frac{\vec{x} \cdot (\vec{v} \times \vec{w})}{\vec{u} \cdot (\vec{v} \times \vec{w})}, \qquad b = \frac{\vec{x} \cdot (\vec{w} \times \vec{u})}{\vec{u} \cdot (\vec{v} \times \vec{w})}, \qquad c = \frac{\vec{x} \cdot (\vec{u} \times \vec{v})}{\vec{u} \cdot (\vec{v} \times \vec{w})}$$

2. **(Lines and Planes)**

   (a) The two lines have no point of intersection and have non-parallel directions.

   (b) $\frac{38}{\sqrt{285}}$

3. **(Curves and Quadric Surfaces)**

   (a) Expand
   $$(x, y, z) \cdot (y, z, x) \leq \|(x, y, z)\|\|(y, z, x)\|$$

   (b) $t\langle 1, 1, 1\rangle$

   (c) Plug equation of L_2, $\langle 1+t, t, t\rangle$, into the equation for S, and check that it is satisfied.

   (d) $\sqrt{\frac{2}{3}}$

   (e) $x + y + z = 0$

   (f) $\sqrt{\frac{2}{3}}$

Problems on Functions of Several Variables

Example 1 (Continuity and Differentiability)

Consider the function

$$f(x, y, z) = \begin{cases} \frac{|x^2 y|}{x^2+y^2+z^2} & (x, y, z) \neq (0, 0, 0) \\ 0 & (x, y, z) = (0, 0, 0) \end{cases}$$

(a) Show f is continuous at $(0, 0, 0)$.

(b) If f differentiable at $(0, 0, 0)$? Justify your answer.

(c) Let a, b, c be positive integers and

$$g(x, y, z) = \begin{cases} \frac{|x^a y^b|}{(x^2+y^2+z^2)^c} & (x, y, z) \neq (0, 0, 0) \\ 0 & (x, y, z) = (0, 0, 0) \end{cases}$$

What conditions do a, b, c have to satisfy for $g(x, y, z)$ to be differentiable at $(0, 0, 0)$?

(d) Determine whether

$$h(x, y, z) = \begin{cases} \frac{|x^{2016} y^{2016}|}{(x^2+y^2+z^2)^{2015}} & (x, y, z) \neq (0, 0, 0) \\ 0 & (x, y, z) = (0, 0, 0) \end{cases}$$

is differentiable at $(0, 0, 0)$

Example 2 (Gradient and Implicit Differentiation)

Consider the surface S given by

$$F(x, y, z) = z^2 x - 2xyz + y^2 = 1$$

(a) Find the tangent plane to S at the point $P = (1, 2, 1)$.

(b) Near the point $P = (1, 2, 1)$, can S be represented as the graph of a function $z = f(x, y)$?

(c) Let $f(x, y)$ be the implicit function from the previous part. For a unit vector $\vec{u} \in \mathbb{R}^2$, the directional derivative of f at $(1, 2)$ in the direction $\vec{u}$ is denoted $D_{\vec{u}} f(1, 2)$. Find a vector $\vec{u}$ such that $D_{\vec{u}} f(1, 2)$ is maximum possible.

(d) Let C be the intersection of the surface S and the plane $y = z$. Find a parametrization of C.

Example 3 (Chain Rule)

(a) Show that

$$f_r = \cos\theta f_x + \sin\theta f_y, \qquad f_\theta = -r\sin\theta f_x + r\cos\theta f_y$$

where $f_r = \frac{\partial f}{\partial r}$, $f_\theta = \frac{\partial f}{\partial \theta}$.

(b) Show that the Laplace equation

$$f_{xx} + f_{yy} = 0$$

in polar coordinates becomes

$$f_{rr} + \frac{1}{r} f_r + \frac{1}{r^2} f_{\theta\theta} = 0, \qquad r > 0$$

(c) Suppose $f(r, \theta) = F(r)$ satisfies the Laplace equation and is independent of θ, find a non-constant example of such a solution $F(r)$ to the Laplace equation.

Example 4 (Quadric Surfaces and Tangent Planes)

Let S be the surface given by $x^2 + y^2 = 4(1 + z^2)$.

(a) Identify the quadric surface S.

(b) $P = (2, 2, 1)$ is a point on S. Find the tangent plane to S at P.

(c) Let C be the intersection of S and the plane Π given by $y = 2z - 10$. Identify the nature of the curve C.

Example 5 (Critical Point Analysis)

For each of the following functions, determine whether $(0, 0)$ is a local min, a local max, a saddle, or none of the above.

(a) $f(x, y) = x^2 + 3xy + 3y^2$

(b) $f(x, y) = x^7 y^7$

(c) $f(x, y) = |x - y| + |x + y| + |xy|$

(d)

$$f(x,y) = \begin{cases} \frac{\sin(x^2+xy+y^2)}{x^2+xy+y^2} & (x,y) \neq (0,0) \\ 1 & (x,y) = (0,0) \end{cases}$$

Example 6 (Constrained Optimization)

Find the global maximum and minimum of the function $f(x, y) = 3x+2y$ subject to the constraint $g(x, y) = \sqrt{x} + \sqrt{y} = 5$.

Answers

1. **(Continuity and Differentiability)**

   (a) Use Sandwich Theorem to show that limit of f exists at $(0, 0, 0)$ exists and equals 0.

   (b) No

   (c) $a + b > 2c + 1$

   (d) Yes

2. **(Gradient and Implicit Differentiation)**

   (a) $3x - 2y + 2z = 1$

   (b) Yes

(c) $\frac{1}{\sqrt{13}}\langle 3, -2\rangle$

(d) $\vec{r}(t) = \left\langle 1 - \frac{1}{t^2}, t, t \right\rangle$, $t \neq 0$.

3. **(Chain Rule)**

   (a) Apply Chain Rule

   $$f_r = f_x x_r + f_y y_r, \qquad f_\theta = f_x x_\theta + f_y y_\theta$$

   (b) Apply Chain Rule and Product Rule

   $$f_{rr} = f_{xx}\cos^2\theta + 2f_{xy}\cos\theta\sin\theta + f_{yy}\sin^2\theta$$

   $$f_{\theta\theta} = f_{xx}r^2\sin^2\theta - 2f_{xy}r^2\cos\theta\sin\theta + f_{yy}r^2\cos^2\theta - rf_x\cos\theta - rf_y\sin\theta$$

   (c) $\ln\sqrt{x^2+y^2}$

4. **(Quadric Surfaces and Tangent Planes)**

   (a) Hyperboloid of one sheet.

   (b) $x + y - 2z = 2$

   (c) Parabola

5. **(Critical Point Analysis)**

   (a) local min

   (b) saddle

   (c) local min

   (d) local max

6. **(Constrained Optimization)**

   f attains maximum value of 75 at $(25, 0)$ and minimum value of 30 at $(4, 9)$.

Problems on Multiple Integrals

Example 1 (Double Integral - Order of Integration)

Evaluate
$$\int_0^1 \int_{y^3}^1 \cos \frac{y}{x^{1/3}} dxdy$$

Example 2 (Triple Integral - Rectangular Coordinates)

x, y, z are randomly chosen from the interval $[1, 3]$. Find the probability that x, y, z are the lengths of the three sides of a triangle.
Hint: Note that for x, y, z to be the lengths of three sides of a triangle, we need
$$x + y \geq z, \qquad y + z \geq x, \qquad z + x \geq y$$
By symmetry, assume x is the largest among the x, y, z, and multiply the result by 3.

Example 3 (Triple Integral - Cylindrical Coordinates)

Find the volume of the solid W given by
$$x^2 + y^2 + z^2 \leq 2z, \qquad x^2 + y^2 + z^2 \leq 1$$

Example 4 (Change of Variable Formula)

Evaluate $\int_0^\infty \int_0^\infty \frac{1}{(1+2x^2-2xy+y^2)^2} dxdy$

Answers

1. **(Double Integral - Order of Integration)**
$$\frac{3}{4} \sin(1)$$

2. **(Triple Integral - Rectangular Coordinates)**
$$\frac{15}{16}$$

3. **(Triple Integral - Cylindrical Coordinates)**

$$\frac{5}{12}\pi$$

4. **(Change of Variable Formula)**

$$\frac{3}{8}\pi$$

Problems on Integration of Vector Fields

Example 1 (Conservative Vector Fields)

Let

$$\vec{r} = \langle x, y, z \rangle, \qquad r = \|\vec{r}\| = \sqrt{x^2 + y^2 + z^2}$$

(a) Show that $\nabla \times (g(r)\vec{r}) = \vec{0}$.You may used the product rule

$$\nabla \times (f\vec{F}) = \nabla f \times \vec{F} + f\nabla \times \vec{F}$$

(b) Define

$$\vec{F}(x, y, z) = \frac{\vec{r}}{r^3} + \frac{1}{r^2}\langle -y, x, 0 \rangle, \qquad \text{where} \qquad (x, y, z) \neq (0, 0, 0)$$

Find $\nabla \times \vec{F}$.

(c) Is the $\vec{F}$ defined above a conservative vector field?

Example 2 (Line Integral)

Let

$$\vec{F} = \left\langle -\frac{y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right\rangle,$$

(a) Let C_1 be the circle $x^2+y^2 = 1$ on the xy-plane oriented counterclockwise. Evaluate $\oint_{C_1} \vec{F} \cdot d\vec{r}$.

(b) Let C_2 be the curve of intersection of the plane $x + y + z = 0$ with the ellipsoid $x^2 + y^2 + \frac{z^2}{3} = 12$. C_2 is an ellipse. Find the lengths of the major and minor axes of C_2.

(c) Find a parametrization of C_2 with counterclockwise orientation when viewed from above.

(d) Evaluate $\oint_{C_2} \vec{F} \cdot d\vec{r}$.

(e) Let C_3 be the part of C_2 that starts from the point $(\sqrt{6}, -\sqrt{6}, 0)$ and ends at the point $(3, 0, -3)$.Evaluate $\oint_{C_3} \vec{F} \cdot d\vec{r}$.

Example 3 (Flux Integral)

Consider the surface S given by $x^2 + \frac{y^2}{4} + z^2 = 1$ to be the boundary of the ellipsoid E enclosed.

(a) If $\vec{F}(x, y, z) = \langle -2xz, e^y + 2, z^2 - e^y z \rangle$, is there a vector field $\vec{G}$ such that $\vec{F} = \nabla \times \vec{G}$?

(b) Let S be oriented inward, and let S_- be the portion of S where $y \leq 0$. Find $\int \int_{S_-} \vec{F} \cdot \vec{n} d\sigma$.

(c) Suppose $T(x, y, z)$ is a harmonic function on E, i.e.

$$T_{xx} + T_{yy} + T_{zz} = 0$$

Show that

$$\int \int_S (D_{\vec{n}} T) T d\sigma = - \int \int \int_E \left(\frac{\partial T}{\partial x} \right)^2 + \left(\frac{\partial T}{\partial y} \right)^2 + \left(\frac{\partial T}{\partial z} \right)^2 dV$$

where $D_{\vec{n}} T$ is the directional derivative of T in the inward point unit normal $\vec{n}$ direction on S.

(d) Suppose T_1, T_2 are two harmonic functions on E, and $T_1 = T_2$ on S, show that $T_1 = T_2$ on E.

Answers

1. **(Conservative Vector Fields)**

   (a)

$$\nabla r = \frac{\vec{r}}{r}, \qquad \nabla \times (g(r)\vec{r}) = g(r)\nabla \times \vec{r} + g'(r)(\nabla r) \times \vec{r} = \vec{0}$$

   (b) $\frac{2z}{r^4}\vec{r}$

   (c) No

2. **(Line Integral)**

   (a) 2π

   (b) $\sqrt{\frac{108}{5}}$ and $\sqrt{12}$

   (c) $\sqrt{\frac{18}{5}} \cos t \, \langle 1, 1, -2\rangle + \sqrt{6} \sin t \langle -1, 1, 0\rangle$

   (d) 2π

   (e) $\frac{\pi}{4}$

3. **(Flux Integral)**

   (a) $\langle e^y z + 2z, xz^2, 0\rangle$

   (b) 3π

   (c) $D_{\vec{n}}T = \nabla T \cdot \vec{n}$, Apply Divergence Theorem to the flux integral

$$\int\int_S T\nabla T \cdot \vec{n} d\sigma$$

   and note that

$$\nabla \cdot (T\nabla T) = T\Delta T + \|\nabla T\|^2$$

   (d) $T_3 = T_1 - T_2$ is harmonic, and

$$\int\int_S T_3 d\sigma = \int\int\int_E \|\nabla T_3\|^2 dV = 0, \qquad \nabla T_3 = \vec{0}$$

   T_3 is constant on E.