MAT 215 Sample Problems

1)

Suppose $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ satisfies the following four properties: (a) f is continuous for $x \geq 0$, (b) $f'(x)$ exists for $x > 0$, (c) $f(0) = 0$, and (d) f' is monotonically increasing. Let $g : \mathbb{R}_{>0} \to \mathbb{R}$ be defined by $g(x) = \dfrac{f(x)}{x}$. Prove that g is monotonically increasing.

2)

Suppose $\{a_n\}_{n=1}^{\infty}$ is a sequence of non-negative terms such that $\displaystyle\sum_{n=1}^{\infty} a_n$ converges. Prove that $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n}$ converges.

3)

Suppose $\{a_n\}_{n=1}^{\infty}$ is a sequence of positive terms such that $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges. Prove that $\displaystyle\sum_{n=1}^{\infty} \frac{a_n}{1+n^2 a_n}$ always converges, and that $\displaystyle\sum_{n=1}^{\infty} \frac{a_n}{1+n a_n}$ can either converge or diverge depending on the choice of $\{a_n\}$.

4)

Let X be a nonempty set and let Y be the set of all subsets of X. Show that there is no bijection from X to Y. (A bijection from X to Y is a function $f : X \to Y$ with the following two properties: (i) for any $a, b \in X$, if $a \neq b$ then $f(a) \neq f(b)$ AND (ii) for any $d \in Y$ there exists at least one $c \in X$ such that $f(c) = d$.)

5)

Let X be a compact metric space with metric $d(x, y)$. Let $A \subseteq X$ be a dense subset. Show that for any $\epsilon > 0$ there exists a finite number elements $a_1, ..., a_n \in A$ such that every $x \in X$ is within ϵ from at least one of the a_i.