Geometry of Homfly-PT

Homfly-PT polynomial

Oriented link $L \subset S^3 \longrightarrow P_L(a,q) \in \mathbb{Q}(a,q)$

Satisfies + is determined by

① Skein relation

$$a\,P(\nearrow\!\!\!\!\nwarrow) - a^{-1} P(\nwarrow\!\!\!\!\nearrow) = (q-q^{-1})\,P(\uparrow\,\uparrow)$$

② Normalization:

$$P(\bigcirc) = 1$$

$$\overline{P}(\bigcirc) = \frac{a-a^{-1}}{q-q^{-1}}$$

unreduced polynomial

Ex:

$$a\,P(\text{Hopf}_+) - a^{-1} P(\text{Hopf}_-) = (q-q^{-1})\,P(\bigcirc)$$

$$\Rightarrow P(\bigcirc\ \bigcirc) = \frac{a-a^{-1}}{q-q^{-1}}$$

$$P(\underbrace{\bigcirc \cdots \bigcirc}_{n}) = \left(\frac{a-a^{-1}}{q-q^{-1}}\right)^{n-1}$$

$$(q-q^{-1})^{|L|-1} P_L \in \mathbb{Z}[a^{\pm 1}, q^{\pm 1}]$$

Specializations:

$a=1 \rightsquigarrow \Delta_L$ Alexander polynomial

$a=q^2 \rightsquigarrow V_L$ Jones polynomial

Hoste
Ocneanu
Millet
Freyd
Lickorish
Yetter
Pryztycki
Traczyk

Goals:

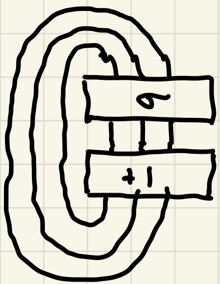
① How to define P_L?

$\rightsquigarrow$ Hecke algebra $\rightsquigarrow$ flag varieties

Kazhdan-Lusztig basis

Categorification H_L Khovanov-Rozansky

② braid closure $L = \overline{\sigma}$



closure $\overline{\sigma}$

Effect of adding full twist to σ ?

J-W projectors → Hilbert schemes of points in $\mathbb{C}^2$

colored polynomials

torus knots $T(n,m)$

③ Rational knots $K_{p/q}$

$K_{13/5}$

$P(K_{p/q}) = ?$

$P^{\wedge^k}(K_{p/q}) = ?$ → $\mathrm{Sym}^k\, D^2$ (Wedrich)

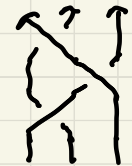
quivers (Sulkowski et al.)

Definition of P_L via Hecke algebra

$\sigma_i = | \cdots \times_i \cdots |$
$\sigma_i^{-1} = | \cdots \times_i \cdots |$

Braid Group

$$Br_n = \left\langle \sigma_1, \dots \sigma_{n-1} \;\middle|\; \underbrace{\begin{matrix} \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \\ \sigma_i \sigma_j = \sigma_j \sigma_i \quad |i-j| > 1 \end{matrix}}_{BR(\sigma)} \right\rangle$$

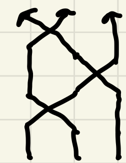


$= \sigma_1 \sigma_2^{-1} \sigma_1^{-1}$

$\downarrow$

Symmetric Group

$$S_n = \left\langle s_1, \dots s_{n-1} \;\middle|\; BR(s),\ s_i^2 = 1 \right\rangle$$



$= s_1 s_2 s_1$

Hecke algebra

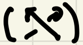
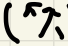
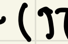
= algebra over $\mathbb{Z}[q^{\pm 1}]$ $\quad (\mathbb{C}(q))$

$$H_n = \left\langle T_1, \dots, T_{n-1} \;\middle|\; BR(T),\ (T_i + q)(T_i - q^{-1}) = 0 \right\rangle$$

$$\sigma_i \longrightarrow T_i \longrightarrow s_i$$

$$\begin{tikzcd} Br_n \arrow[r, "\psi"] \arrow[d] & H_n \arrow[dl, "q=1"] \\ \mathbb{Z} S_n & \end{tikzcd}$$

$$T_i^2 + (q - q^{-1})T_i - 1 = 0 \quad \Rightarrow \quad T_i^{-1} - T_i = (q - q^{-1}) \cdot 1$$

$\psi($  $) - \psi($  $) = (q - q^{-1})\, \psi($  $)$

skein relation

Alternate generators:

$B_i =$ 

$$B_i = q^{-1} - T_i \qquad T_i = q^{-1} - B_i$$

$$B_i = q - T_i^{-1} \qquad T_i^{-1} = q - B_i$$

$$B_i^2 = (q + q^{-1}) B_i$$

$$H_n = \left\langle B_1, \ldots B_{n-1} \;\middle|\; \begin{array}{l} B_i B_j = B_j B_i \quad |i-j| > 1 \\ B_i B_{i+1} B_i - B_i = B_{i+1} B_i B_{i+1} - B_{i+1} \\ B_i^2 = (q + q^{-1}) B_i \end{array} \right\rangle$$

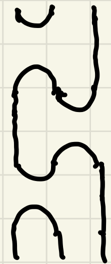
Temperly-Lieb Algebra:

$$TL_n = \left\langle U_1, \dots, U_{n-1} \;\middle|\; \begin{array}{l} U_iU_j = U_jU_i \quad |i-j|>1 \\ U_iU_{i\pm1}U_i = U_i \\ U_i^2 = (q+q^{-1})\,U_i \end{array} \right\rangle$$

Basis = {crossingless planar diagrams}

$$U_i = \|\;\cdots \underset{i}{\stackrel{\cup}{\cap}} \cdots\;\|$$

$U_i U_{i\pm1} U_i - U_i = 0$

$U_1U_2U_1 =$  $= \stackrel{\cup}{\cap}\,| = U_1$

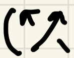
$U_1^2 =$  $= (q+q^{-1})\stackrel{\cup}{\cap}$

$$\begin{tikzcd} Br_n \arrow[r, "\psi"] \arrow[rr, bend left, "\psi_2"] & H_n \arrow[r] & TL_n \end{tikzcd}$$

$\sigma_i \longrightarrow q^{-1} - B_i \longrightarrow q^{-1} - U_i$

$\sigma_i^{-1} \longrightarrow q - B_i \longrightarrow q - U_i$

$B_i \longmapsto U_i$

$\psi_2($$) = q^{-1}\,\|\, - \stackrel{\cup}{\cap}$

$\psi_2($$) = q\,\|\, - \stackrel{\cup}{\cap}$

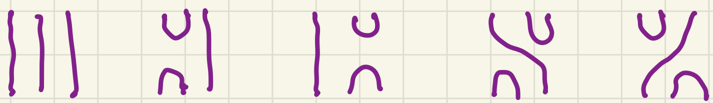
Kauffman bracket skein relation

Product of U_i's $\rightarrow$ crossingless planar diagram

isotopic diagrams $\Rightarrow$ products equal in TL_n

$\Rightarrow$ $\{$crossingless planar (n,n) diagrams$\}$ is a basis for TL_n

Ex: Basis for TL_3 is

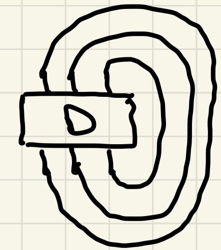


$1 \qquad U_1 \qquad U_2 \qquad U_1U_2 \qquad U_2U_1$

$\dim TL_3 = 5$

Define $Tr : TL_n \to \mathbb{Z}[q^{\pm 1}]$

$$Tr(D) = (q+q^{-1})^{|\bar{D}|}$$



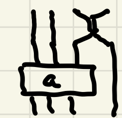
closure $\bar{D}$

Then $\langle \bar{\sigma} \rangle \sim Tr\, \psi_2(\sigma)$ $\qquad |\bar{D}| = \#$ cpts of $\bar{D}$

Kauffman bracket

Jones-Ocneanu Trace:

$c(a) =$ 

$c(a)B_n =$ 

Thm: $\exists$ unique $\mathbb{Z}[q^{\pm 1}]$ linear maps

$Tr_n : H_n \to \mathbb{Z}[a^{\pm 1}, q^{\pm 1}, (q-q^{-1})^{-1}]$

satisfying

1) $Tr_n\, ab = Tr_n\, ba$
2) $Tr_{n+1}\, c(a) = \{0\}\, Tr_n\, a$
3) $Tr_{n+1}\, c(a) B_n = \{1\}\, Tr_n\, a$
4) $Tr_1\, 1 = 1$

$c : H_n \hookrightarrow H_{n+1}$

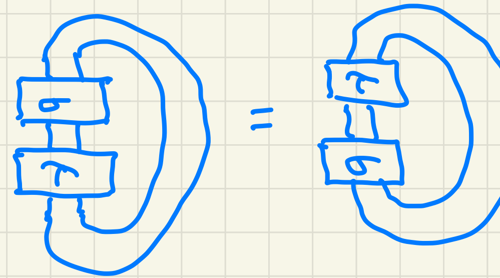
$$\{n\} = \frac{a q^{-n} - a^{-1} q^{n}}{q - q^{-1}}$$

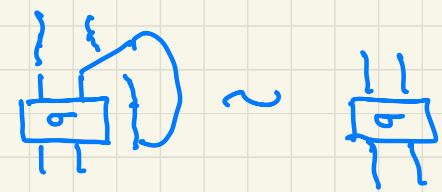
Markov's Thm: If $\overline{\sigma} = \overline{\tau}$, σ and τ are related by a sequence of moves of types I and II

I) $\sigma \longleftrightarrow \tau\sigma\tau^{-1}$

II) $\sigma \longleftrightarrow \iota(\sigma)\sigma_n^{\pm 1}$

$\iota : Br_n \hookrightarrow Br_{n+1}$





$\boxed{\textbf{Def:}\ P(\sigma) = a^{w(\sigma)}\, \mathrm{Tr}\, \Psi(\sigma)}$

Then $P(\tau\sigma\tau^{-1}) = a^{w(\sigma)}\, \mathrm{Tr}\, \tau\sigma\tau^{-1}$
$= P(\sigma)$

$P(\iota(\sigma)\sigma_n) = a(q^{-1}\{0\} - \{1\})P(\sigma) = P(\sigma)$

$P(\iota(\sigma)\sigma_n^{-1}) = a^{-1}(q\{0\} - \{1\})P(\sigma) = P(\sigma)$

$\Rightarrow$ If $\overline{\sigma} = \overline{\tau}$, then $P(\sigma) = P(\tau)$.

$w(\sigma)$ = writhe of σ
$= |\sigma|$ $\qquad |\cdot| : Br_n \to \mathbb{Z}$
$\sigma_i \longmapsto 1$

Reduced Words:

$w = w_1 \dots w_k$ word in $1 \dots n-1$

$s(w) = s_{w_1} \dots s_{w_k} \in S_n$

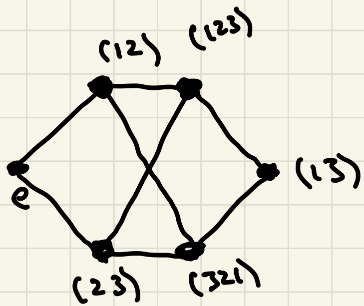
Ex: $s(121) = s_1 s_2 s_1 = s(212)$

If $s \in S_n$, $\ell(s) = \min\{\ell(w) \mid s(w) = s\}$

= # crossings in a minimal diagram of s

$= \#\{(i,j) \mid i<j,\ s(i) > s(j)\}$

Ex. S_3



$\ell =$ 0 1 2 3

Given $s \in S_n$, want a preferred w with $s = s(w)$ $\ell(w) = \ell(s)$

Standard choice = reduced word

Move all crossings w/ nth strand above crossings w/o nth strand

repeat.

Ex: $1 \to 4$, $2 \to 3$, $3 \to 1$, $4 \to 2$

$w = 12132$

If $w \in S_n$ is reduced, $n-1$ appears 0 or 1 time.

Basis for H_n:

Let $X_k = \langle s(w) \mid \ell(w) \leq k \rangle \subset \mathbb{Z}[S_n]$

$Y_k = \langle B(w) \mid \ell(w) \leq k \rangle \subset H_n$

$X_i \cdot X_j \subset X_{i+j}$

Associated graded rings are $\simeq$

$Y_i \cdot Y_j \subset Y_{i+j}$

$\{ s(w) \mid w \text{ is reduced}, \ell(w) \leq k \}$ is a basis for X_k

$\Rightarrow \{ B(w) \mid w \text{ is reduced}, \ell(w) \leq k \}$ is a basis for Y_k

$\{ B(w) \mid w \text{ is reduced} \}$ is a basis for H_n

$\dim H_n = \dim \mathbb{Z}[S_n] = n!$

Construction of Tr_n:

Given Tr_n exists and is unique, show same holds for Tr_{n+1}

1) **Uniqueness.** Tr_{n+1} is determined by

$Tr_{n+1} B(w)$ w reduced. B_n appears 0 or 1 time

$$\begin{array}{ll} Tr_{n+1}\, C(x) & Tr_{n+1}\, C(x) B_n C(y) \\ \parallel & \parallel \\ \{0\}\, Tr_n\, x & \{1\}\, Tr_n\, xy \end{array}$$

2) **Existence.** Must check $Tr_{n+1}\, xy = Tr_{n+1}\, yx$

Enough to show $Tr_{n+1}\, xB_i = Tr_{n+1}\, B_i x$ Easy if $i<n$ or $x \in H_{n-1}$

Interesting case: $Tr_{n+1}(aB_n bB_n) = Tr_{n+1}(B_n a B_n b)$
$a, b \in H_n$

$a \in H_{n-1}$ or $a = a_1 B_{n-1} a_2$ $\quad$ $b \in H_{n-1}$ or $b = b_1 B_{n-1} b_2$ $\rightsquigarrow$ 4 cases

We'll do:

$a \in H_{n-1}$ or $a = a_1 B_{n-1} a_2$

$b \in H_{n-1}$ or $b = b_1 B_{n-1} b_2$

$$\mathrm{Tr}\ a B_n b_1 B_{n-1} b_2 B_n$$
$$= \mathrm{Tr}\ a b_1 B_n B_{n-1} B_n b_2$$
$$= \mathrm{Tr}\ a b_1 (B_{n-1} B_n B_{n-1} - B_{n-1} + B_n) b_2$$
$$= \left(\{1\}^2 [2] - \{0\}\{1\} + \{0\}\{1\}\right) \mathrm{Tr}\ a b_1 b_2$$

$$\mathrm{Tr}\ B_n a B_n b_1 B_{n-1} b_2$$
$$= \mathrm{Tr}\ [2]\, a B_n b_1 B_{n-1} b_2$$
$$= [2]\{1\}^2\, \mathrm{Tr}\ a b_1 b_2$$

$$[2] = q + q^{-1}$$

Bonus Slide:

MFW inequality!

By induction, if $B(w) \in H_n$

$\max_a \operatorname{Tr} B(w) \le n-1$

$\min_a \operatorname{Tr} B(w) \ge 1-n$

$\Rightarrow \quad \max_a P(\bar{\sigma}) \le w+n-1$

$\min_a P(\bar{\sigma}) \ge w+1-n$

$2n \ge \max_a P(\bar{\sigma}) - \min_a P(\bar{\sigma}) + 2$

Lower bound on braid index

$\deg_t \Delta_K(t) \le g(K)$

$\deg_q P_K \le g_{can}(K)$

g_{can} = minimal genus obtained by Seifert's algorithm.

Afterwards: Careful definition of P_2

Given L, choose σ with $\overline{\sigma} = L$. (σ exists by Alexander's Thm.)

$$\boxed{\underline{\text{Def}}: \; P_2 = a^{w(\sigma)} \operatorname{Tr} \psi(\sigma) = P(\sigma)}$$

Must check $P(\sigma) = P(\tau)$ if $\overline{\sigma} = \overline{\tau}$

By Markov's Thm, enough to check

1) $P(\tau\sigma\tau^{-1}) = P(\sigma)$

2) $P(\iota(\sigma)\sigma_n^{\pm 1}) = P(\sigma)$

1) $w(\tau\sigma\tau^{-1}) = w(\sigma)$, so

$$\begin{aligned} P(\tau\sigma\tau^{-1}) &= a^{w(\tau\sigma\tau^{-1})} \operatorname{Tr} \psi(\tau\sigma\tau^{-1}) \\ &= a^{w(\sigma)} \operatorname{Tr} \psi(\sigma\tau^{-1}\tau) \\ &= a^{w(\sigma)} \operatorname{Tr} \psi(\sigma) \\ &= P(\sigma) \end{aligned}$$

2) $P(c(\sigma)\sigma_n) = a^{w(\sigma\sigma_n)} \operatorname{Tr} \psi(c(\sigma)\sigma_n)$

$$= a^{w(\sigma)+1} \operatorname{Tr}\left[c(\psi(\sigma))(q^{-1} - B_n)\right]$$

$$= a^{w(\sigma)+1} \left[q^{-1}\{0\} - \{1\}\right] \operatorname{Tr} \psi(\sigma)$$

$$= \left[a(q^{-1}\{0\} - \{1\})\right] a^{w(\sigma)} \operatorname{Tr} \psi(\sigma)$$

$$= a^{w(\sigma)} \operatorname{Tr} \psi(\sigma) = P(\sigma)$$

Calculation for $P(c(\sigma)\sigma_n^{-1})$ is very similar.

<u>Skein relation</u>: Follows from $\psi(\sigma_i^{-1}) - \psi(\sigma_i) = (q - q^{-1})\psi(1)$