Flags, K-L Basis

+ Categorification

Last Time: Hecke algebra

$$H_n = \left\langle B_1, \ldots B_{n-1} \;\middle|\; \begin{array}{l} B_iB_j = B_jB_i \quad |i-j|>1 \\ B_iB_{i+1}B_i - B_i = B_{i+1}B_iB_{i+1} - B_{i+1} \\ B_i^2 = (q+q^{-1})B_i \end{array} \right\rangle$$

Symmetry $c: H_n \to H_n$, $c(q) = q^{-1}$, $c(B_i) = B_i$

$\{B(w) \mid w \text{ is a reduced word}\}$ is a basis for H_n

Ex: $H_3 = \langle 1, B_1, B_2, \underset{\substack{\| \\ B_{12}}}{B_1B_2}, \underset{\substack{\| \\ B_{21}}}{B_2B_1}, B_1B_2B_1 \rangle$

Better choice of last vector

$$B_{121} = B_1B_2B_1 - B_1 = B_2B_1B_2 - B_2$$

$$B_1B_{121} = B_{121}B_1 = B_2 \cdot B_{121} = B_{121}B_2 = [2]B_{121}$$

$\Rightarrow \langle B_{121} \rangle$ is a 2-sided ideal

$\langle B_{121} \rangle = \operatorname{Ker} \psi: H_3 \to TL_3$

$H_3 = \langle B_s \mid s \in S_3 \rangle$ $\quad 1 = B_e \quad B_w = B_{s(w)}$

<u>Good Properties:</u>

① Filtration by ideals

$$(B_{121}) \subset (B_1, B_2) \subset H_3$$

② $B_s \cdot B_{121} = P(s) B_{121}$

$$\boxed{\begin{aligned} &\overline{Tr}_1\, 1 = \{0\} \\ &\overline{Tr}_{n+1}\, \iota(x) = \{0\}\, Tr_n\, x \\ &\overline{Tr}_{n+1}\, \iota(x) B_n = \{1\}\, Tr_n\, x \end{aligned}}$$

where $[n] = \dfrac{q^n - q^{-n}}{q - q^{-1}}$

$$\{n\} = \frac{a q^{-n} - a^{-1} q^n}{q - q^{-1}}$$

s	$P(s)$	$\overline{Tr}(B_s)$
e	1	$\{0\}^3$
$1, 2$	$[2]$	$\{0\}^2\{1\}$
$12, 21$	$[2]^2$	$\{0\}\{1\}^2$
121	$[2][3]$	$\{0\}\{1\}\{2\}$

<u>Ex:</u> $Tr\, B_{121} = Tr\, B_1 B_2 B_1 - Tr\, B_1$

$$= Tr\, B_1^2 B_2 - Tr\, B_1$$

$$= [2]\{0\}\{1\}^2 - \{0\}^2\{1\}$$

$$= \{0\}\{1\}\big([2]\{1\} - \{0\}\big)$$

$$= \{0\}\{1\}\{2\}$$

Flag Varieties:

V = vector space over $\mathbb{C}$

$Fl(V) = \{ \{0\} = V_0 \subset V_1 \subset \ldots \subset V_n = V \mid \dim V_i = i \}$

$Fl_n = Fl(\mathbb{C}^n)$

$Gr(1,n) \cong \mathbb{C}P^{n-1}$

$\cong$

$\pi : Fl_n \to Gr(n-1,n)$ $\qquad \pi^{-1}(V') = Fl(V') \cong Fl_{n-1}$

$(V_i) \mapsto V_{n-1}$

Fibration
$$\begin{array}{ccc} Fl_{n-1} & \to & Fl_n \\ & & \downarrow \\ & & \mathbb{C}P^{n-1} \end{array}$$
$\Longrightarrow$ Fl_n is a cell cx with $n!$ cells, all even dim'l

Poincare polynomial $\quad \mathcal{P}(Fl_n) = \mathcal{P}(Fl_{n-1})\,\mathcal{P}(\mathbb{C}P^{n-1})$

$= \mathcal{P}(Fl_{n-1})\,[\tilde{n}]$ where $[\tilde{n}] = \dfrac{t^{2n}-1}{t^2-1}$

$\Longrightarrow \boxed{\mathcal{P}(Fl_n) = [\tilde{n}]!}$

$[\tilde{n}]! = [\tilde{1}][\tilde{2}]\cdots[\tilde{n}]$

Bruhat cells:

$G = GL_n(\mathbb{C})$ $\quad$ $B = \begin{Bmatrix} \text{Upper triangular} \\ \text{matrices} \end{Bmatrix} \subset G$

Borel subgroup

$G/B \simeq Fl_n$

$(w_1, \dots w_n) \longmapsto (V_i)$ $\quad$ $V_i = \langle w_1, \dots, w_i \rangle$

column matrix $\quad$ flag

Prop: $\boxed{G = \coprod_{s \in S_n} BsB}$ $\quad$ $\overset{\text{Weyl group}}{\overset{/\!/}{S_n}} \hookrightarrow GL_n(\mathbb{C})$ $\quad$ permutation matrices

$s(e_i) = e_{s(i)}$

Def: V_s = image of BsB in $G/B = Fl_n$ $\quad$ Bruhat cell

Prop: V_s is a cell of dim $2\ell(s)$

Proof: If $b_1, b_2 \in B$, $b_1 A b_2$ is obtained from A by multiplying rows, columns by constants adding lower rows to upper, left columns to right.

Given $A \in GL_n(\mathbb{C})$, reduce to a permutation matrix by

1) Find first nonzero entry in bottom row.
2) Multiply to make it 1.
3) Clear all entries in column above and row to right
4) Repeat w/ next row up.

Ex:
$$\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \to \begin{pmatrix} * & 0 & * \\ 0 & 0 & * \\ 0 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} * & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

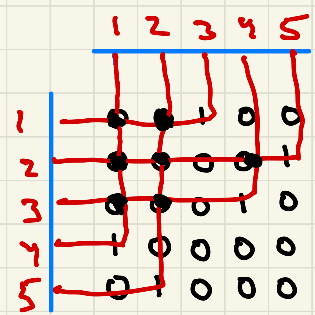
Picture Proof of ②:

If $s = \begin{matrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{matrix}$ Flags in V_s look like $\begin{matrix} \bullet & \bullet & 1 & \bullet & \bullet \\ \bullet & \bullet & 0 & \bullet & 1 \\ \bullet & \bullet & 0 & 1 & 0 \\ 1 & \bullet & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{matrix}$ 0's below 1's

Any such flag can be uniquely expressed as $\begin{matrix} \bullet & \bullet & 1 & 0 & 0 \\ \bullet & \bullet & 0 & \bullet & 1 \\ \bullet & \bullet & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{matrix}$ 0's to right of 1's

\# of free variables ($\bullet$'s) =

\# of crossings in this string diagram for s



Bruhat Order on S_n:

$s' < s$ if we can get from a minimal length string diagram for s to a string diagram for s' by resolving crossings.

$s' < s \Rightarrow \ell(s') < \ell(s)$

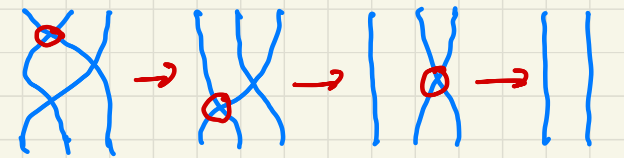
Prop: $V_{s'} \subset \overline{V_s} \iff s' < s$

Cor: $\sum_{s \in S_n} t^{\ell(s)} = \widetilde{[n]}!$

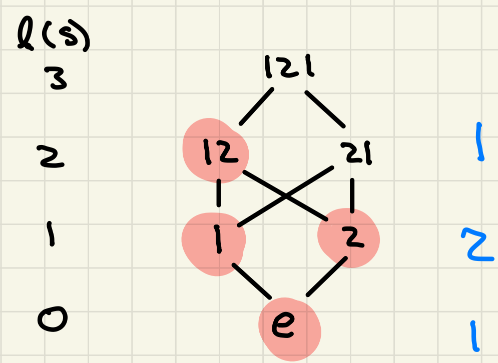
Ex: $n=3$

s	$\mathcal{P}(\overline{V_s}) = P(s)$
e	1
$1,2$	$[2]$
$12,21$	$[2]^2$
121	$[2][3]$

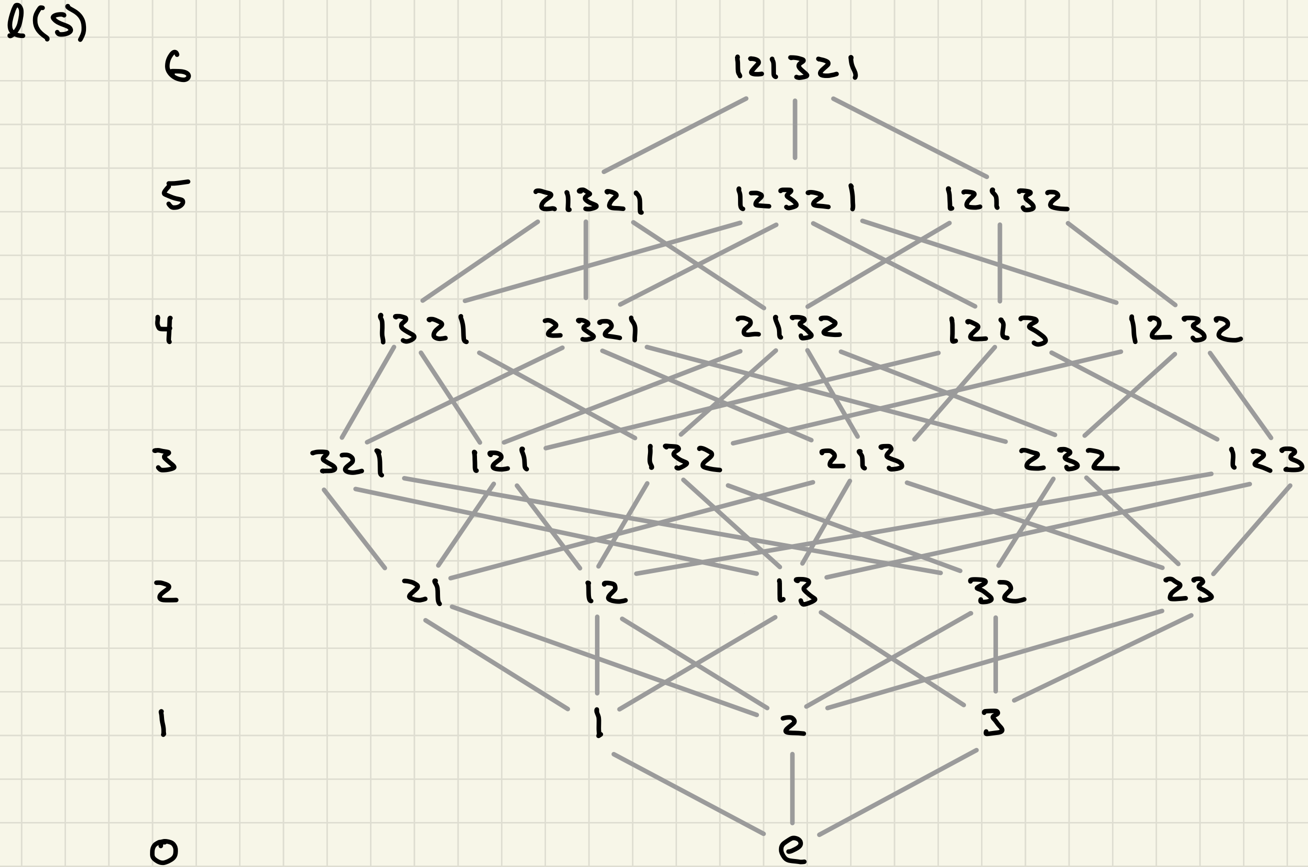
Ex: S_3 $\quad 121 > 12 > 2 > e$



Bruhat diagram for S_3 / Fl_3



Bruhat diagram for S_4 / Fl_4



Kazhdan-Lusztig Basis:

Basis $\{B_s \mid s \in S_n\}$ of H_n

Properties:

① Length Filtration

If $s = s_i s'$, $\ell(s) > \ell(s')$

$$B_i B_{s'} = B_s + \sum_{\ell(t) < \ell(s')} c_t B_t$$

② Symmetry: $\iota(B_s) = B_s$

$\iota : H_n \to H_n$, $\iota(B_i) = B_i$, $\iota(q) = q^{-1}$

③ Filtration by Ideals

If $\mu \perp n$, $I_\mu = \langle B_s \mid \lambda(s) \geq \mu \rangle$

is a 2-sided ideal in H_n

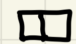
Partitions:

$\lambda = \{\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k > 0\}$

is a partition of $n = \sum \lambda_i$

write $\lambda \perp n$

Represent by Young diagrams

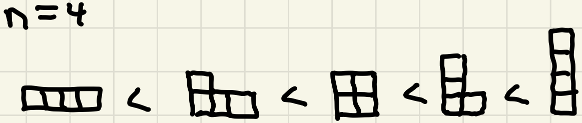
 $= \{2\}$  $= \{1,1\}$

 $= \{2,1\}$

Partial order: $\lambda, \mu \perp n$

$\lambda \geq \mu$ if $\sum_{i=1}^{k} \lambda_i \geq \sum_{i=1}^{k} \mu_i$ for all k

Ex: $n=4$



Robinson-Schensted:

$s \in S_n \longrightarrow \lambda(s) \vdash n$

Insert $s(1), s(2), \ldots, s(n)$ to build Young tableau.

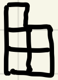
To insert k into T:

1) Look at bottom row
2) If $k >$ all elements, add k to end. Stop.
3) If not, let k' be smallest entry $> k$
4) Replace k' with k. Repeat 1) with k' and next row up.

Ex: $n = 5$

$$s: \begin{array}{l} 1 \to 4 \\ 2 \to 5 \\ 3 \to 1 \\ 4 \to 3 \\ 5 \to 2 \end{array}$$

$$4 \to 4\ 5 \to \begin{array}{ll} 4 & \\ 1 & 5 \end{array} \to \begin{array}{ll} 4 & 5 \\ 1 & 3 \end{array} \to \begin{array}{ll} 4 & \\ 3 & 5 \\ 1 & 2 \end{array}$$

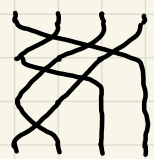
$\Rightarrow \lambda(s) =$ 

Longest word w_0 is the only $s \in S_n$ with $\lambda(s) =$  $= \{n\}$

$w_0(i) = n - i$

$\ell(w_0) = \binom{n}{2}$

Ex: $n = 4$



$w_0 = 121321$

$\lambda(S)$ shown on Bruhat diagram for S_4

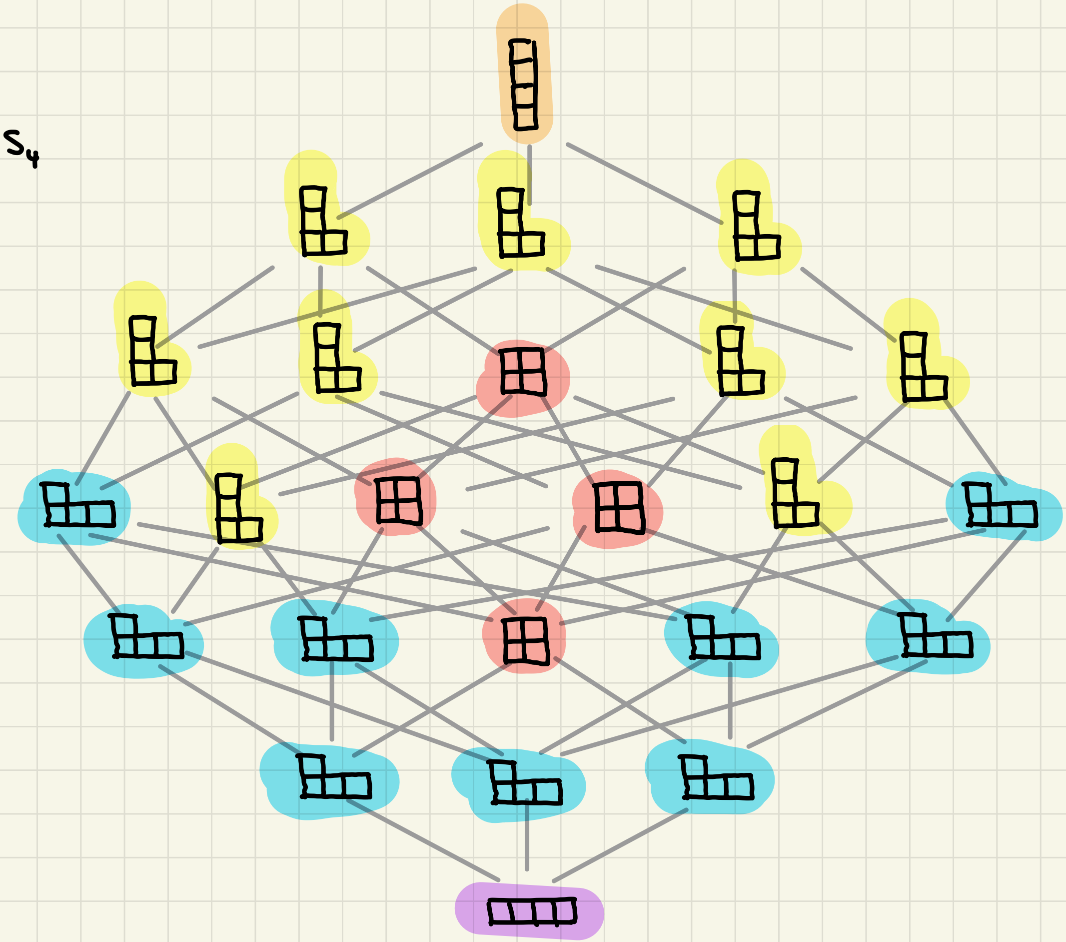
1

9

4

9

1



④ Smallest Ideal

$I_{\{n\}} = \langle B_{w_0} \rangle$

$\Rightarrow B_s \cdot B_{w_0} = P_s(q) B_{w_0}$

What is $P_s(q)$?

By symmetry $P_s(q) = P_s(q^{-1})$

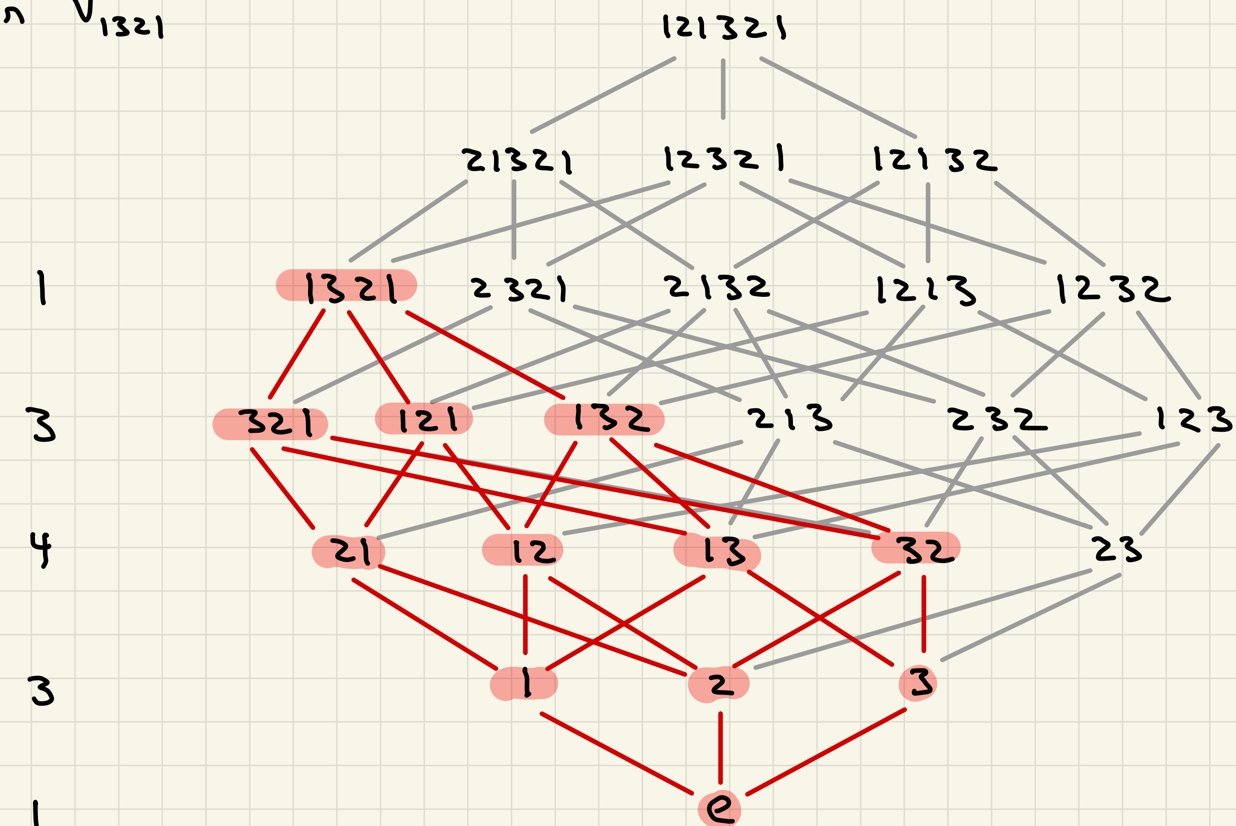
<u>$n=3$</u>: $P_s(q) \sim P(\overline{V}_s)$

$\overline{V}_s$ = Schubert variety

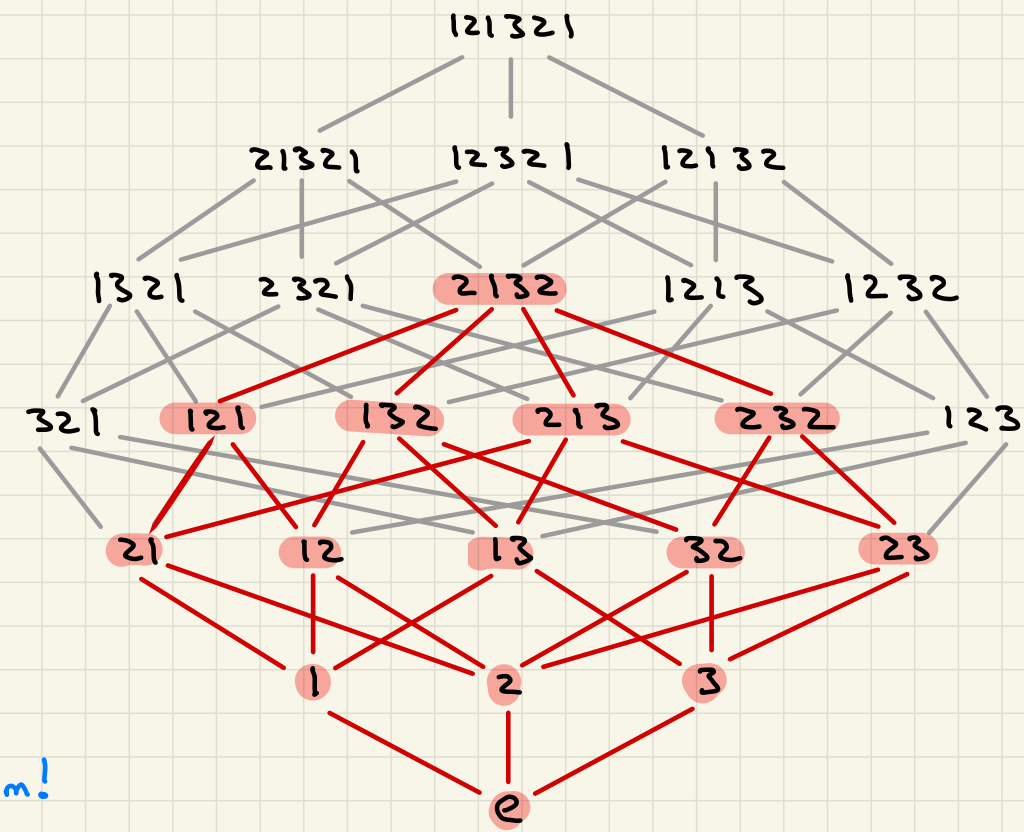
<u>$n=4$</u>: works for most s, e.g.

$$P(\overline{V}_{1321}) \sim [2]^2[3] = P_{1321}(q)$$

cells in $\overline{\nabla}_{1321}$



Cells in $\overline{V}_{2132}$



1

4

5

3

1

Problem!

④ Smallest Ideal

$I_{\{n\}} = \langle B_{w_0} \rangle$

$\Rightarrow B_s \cdot B_{w_0} = P_s(q) B_{w_0}$

What is $P_s(q)$?

By symmetry $P_s(q) = P_s(q^{-1})$

<u>n=3</u>: $P_s(q) \sim \mathcal{P}(\bar{V}_s)$

$\bar{V}_s$ = Schubert variety

<u>n=4</u>: works for most s, e.g.

$$\mathcal{P}(\bar{V}_{1321}) \sim [2]^2[3] = P_{1321}(q)$$

$\bar{V}_{1321}$ is smooth

But $\mathcal{P}(\bar{V}_{2132}) = 1 + 3t^2 + 5t^4 + 4t^6 + t^8$

is not symmetric

$\bar{V}_{2132}$ is not; does not satisfy Poincaré duality

<u>Solution</u>: $P_s(q) \sim \mathcal{P}(IH^*(\bar{V}_s))$

IH^* = Intersection cohomology (Goresky–Macpherson)

$$\mathcal{P}(IH^*(\bar{V}_{2132})) = [2]^4$$

⑤ Positivity

$$B_s B_{s'} = \sum_{s''} c_{s,s'}^{s''} B_{s''}$$

$c_{s,s'}^{s''} \in \mathbb{N}[q^{\pm 1}]$!! (KL-Conjecture)

Why?

<u>Hecke Category $\mathcal{H}_n$</u> : additive monoidal category additively generated by objects $\mathbb{B}_s$ ($s \in S_n$)

with $\mathbb{B}_s \mathbb{B}_{s'} = \bigoplus_{s''} c_{s,s'}^{s''} \mathbb{B}_{s''}$

<u>Models for $\mathcal{H}_n$</u>:

- Perverse sheaves on Fl_n
  (Beilinson-Bernstein / Kashiwara)

- Bimodules over $R_n = \mathbb{C}[x_1, \dots x_n]$
  (Soergel)

Soergel Bimodules:

S_n acts on R_n by permuting x_i's

$R_n^{\langle s_i \rangle}$ = ring of invariants $= \mathbb{C}[x_1, \dots x_{i-1}, e_1, e_2, x_{i+2}, \dots x_n]$

$\mathbb{B}_i = R \otimes_{R^{s_i}} R$ $\qquad e_1 = x_i + x_{i+1} \quad e_2 = x_i x_{i+1}$

$SBim_n$ = full subcategory of R_n-R_n bimodules

generated by $\mathbb{B}_i$, taking $\otimes$, direct summands

Ex: As left module over $R_n^{s_i}$, $R_n = R_n^{s_i} \oplus x_1 R_n^{s_i} \sim (1+q^2) R_n^{s_i}$

free $R_n^{s_i}$ module generated by 1, x_1

$$\Rightarrow \mathbb{B}_i \otimes \mathbb{B}_i \simeq R \otimes_{R^i} (1+q^2) R^i \otimes_R R \otimes_{R^i} R$$

$$\simeq (1+q^2) R \otimes_{R^i} R^i \otimes_{R^i} R \sim (q+q^{-1}) \mathbb{B}_i$$

$$\mathbb{B}_{w_0} = R \otimes_{R^{S_n}} R$$

Hochschild Homology:

Functor $HH: R\text{-mod-}R \to R\text{-mod}$

$\widetilde{R} = R[x_i, x_i']$ $\quad R\text{-mod-}R \leftrightarrow \widetilde{R}\text{-mod}$

$HH_*(\mathbb{B}) = \mathrm{Tor}_*^{\widetilde{R}}(\mathbb{B}, R)$

Prop: (Khovanov) If $\mathbb{B} \in SBim_n$

$$\hat{P}(HH_*(\mathbb{B})) = \overline{Tr}\,[\mathbb{B}]$$

$$\sum (-a)^i \, \mathrm{qdim}\, HH_i(\mathbb{B})$$

Not $\chi(HH_*(\mathbb{B}))$

$[\mathbb{B}_s] = B_s$

Proof: Check properties 1) - 4) of Tr.

Geometry:

Soergel: $\mathbb{B}_s = IH^*_{B\times B}(\overline{G}_w)$

$\overline{G}_s = \overline{BsB} \subset G$

$\shortparallel$

$\pi^{-1}(\overline{V}_s)$

Equivariant cohomology:

B acts on left and right

$H^*_B(pt) \simeq H^*_T(pt) \simeq R_n$

$B \sim T$ maximal torus

Thm: (Williamson-Webster) $HH_*(\mathbb{B}_s) \simeq IH^*_B(\overline{G}_s)$

B acts by conjugation

$\overline{V}_s$ smooth $\Rightarrow IH^*_B(\overline{G}_s) \simeq H^*_B(pt) \otimes H^*(\overline{G}_s)$

$\overline{V}_s$ = iterated projective bundle

$\overline{G}_s$ = iterated sphere bundle.

Ex: $H^*(\overline{G}_{w_0}) = H^*(GL_n(\mathbb{C}))$

$= H^*(U(n))$

$= \Lambda^*(x_1, x_3, \dots x_{2n-1})$